Mechanism Design for Correlated Valuations: Efficient Methods for Revenue Maximization

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Traditionally, mechanism design has been primarily focused on settings where the valuations for individual bidders are drawn independently. However, in settings where valuations are *correlated*, much stronger results are possible. For example, the entire surplus of efficient allocation can be extracted as revenue. While these much stronger results are possible with generic conditions, they are rarely, if ever, achieved in practice due to strong requirements that the mechanism designer knows the distribution of bidders exactly. In this work, we provide a both computationally and sample efficient method to design mechanisms that can robustly incorporate an imprecise estimate of the distribution over bidder valuations, using samples from the true distribution, in a way that provides strong guarantees that the mechanism will perform at least as well as ex-post mechanisms, while also performing nearly optimally with sufficient information. Additionally, we provide, for the first time, necessary and sufficient conditions for full surplus extraction as revenue in a correlated valuation setting, and we characterize the set of problems for which learning the optimal mechanism will be feasible. We also demonstrate through simulation that this new mechanism design paradigm generates mechanisms that perform significantly better than traditional mechanism design techniques.

Key words: mechanism design; robust optimization; revenue maximization; correlated valuations; scoring rules

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1. Introduction

Auctions are one of the fundamental tools of the modern economy for allocating resources. They are used to allocate online ad space, offshore oil drilling rights, famous artwork, small and medium lift capacity to planetary orbit, government supply contracts, FCC spectrum licenses, and almost limitless numbers of other things, large and small. Further, the sizes of these markets are economically enormous. In 2016, $72.5 billion dollars of ad revenue was generated through automated auctions (Interactive Advertising Bureau (IAB) 2017). In 2012, just four government agencies—the Army, the Department of Homeland Security, the Department of the Interior, and the Department of Veteran Affairs—purchased $800+ million of commercial items through auctions (Government Accountability Office 2013). In 2014, NASA awarded contracts to Boeing and Space-X worth $4.2 billion and $2.6 billion respectively through an implicit auction process (NASA 2014). In the most recent spectrum auction, the FCC allocated $20 billion worth of broadcast spectrum (FCC 2017). Given the economic magnitudes involved, it is crucial that these auctions are implemented optimally, for even small deviations from optimality can lead to millions of dollars worth of lost revenue, inefficiencies in resource allocation, and overspending.

It has long been understood that revenue optimal auction mechanisms are *prior-dependent* (also known as *Bayesian*) mechanisms (Myerson 1981, Cremer and McLean 1985, 1988, Lopomo 2001), i.e., mechanisms that assume some knowledge of the kinds of bidders that are likely to participate. For a seller trying to maximize revenue by selling a single item to symmetric independent bidders, she (in this paper, we will use "he" to denote bidders and "she" to denote mechanism designers/sellers) needs to set a reserve price below which she will not sell the item (Myerson 1981), and this reserve price is dependent on her belief about the bidders she is likely to face.

However, much of the focus of the mechanism design community has been on the approximate optimality of *simple* mechanisms (Bulow and Klemperer 1996, Hartline and Roughgarden 2009, Roughgarden and Talgam-Cohen 2013, Morgenstern and Roughgarden 2015), that is, mechanisms that are either prior-independent or weakly prior-dependent (this distinction will be made clear later on). This focus is due to two factors. First, prior-dependent mechanisms can be very brittle to mis-specified priors (Lopomo 2001, Albert et al. 2015). That is to say, if a prior-dependent mechanism is constructed using an incorrect prior it can perform much worse than simple mechanisms (Hartline and Roughgarden 2009). Second, competition can be an effective substitute for knowledge of the distribution (Bulow and Klemperer 1996), so instead of implementing prior-dependent mechanisms, practitioners generally implement prior-independent mechanisms under the assumption that there are many bidders, or that it will somehow be possible to *acquire* new bidders at a reasonable cost.

Unfortunately, in many auctions there is no feasible way to acquire more bidders. When NASA is awarding contracts to private space companies, they cannot generate more companies with the expertise to provide lift capacity, nor can Google reasonably find more bidders who are interested in advertising on search results for obscure brand-related keywords. Thin auctions are a well recognized concern for many organizations that use auction mechanisms. A Government Office of Accountability report from 2013 (Government Accountability Office 2013) examining the use of reverse auctions by four governmental organizations found that of the 19,688 reverse auctions the organizations conducted in 2012 with a total worth of $800+ million, over one-third had only a *single bidder*. Further, the organizations discussed in the report each voiced concern over the lack of competition as a significant hindrance to the effective application of auctions.

Therefore, to generate effective outcomes from the implementation of mechanisms in thin markets, a mechanism designer must both learn and incorporate information about the distribution of bidders into the design of the mechanism, and a recent literature (Pardoe et al. 2010, Fu et al. 2014, Kanoria and Nazerzadeh 2014, Mohri and Medina 2014, Blum et al. 2015, Mohri and Medina 2015, Morgenstern and Roughgarden 2015) has begun to address this particular challenge by developing techniques to optimally learn the prior distribution and construct mechanisms given samples from the true distribution. However, the literature has, primarily, focused on the restrictive case of independent private value (IPV) distributions, where each bidder's valuation is independent of all other bidders, and the restrictive class of ex-post individually rational and dominant strategy incentive compatible mechanisms.

In the more general setting of correlated valuation distributions, i.e. settings where one bidder's valuation is correlated with other bidders' valuations, and fully Bayesian mechanisms, where the beliefs of one bidder for other bidders affects his strategy, much less is known. Moreover, the combination of correlated valuations and Bayesian mechanisms is unique in that it allows for the strongest possible result in revenue maximizing mechanism design, that of full surplus extraction as revenue for the seller (Cremer and McLean 1985, 1988). Essentially, with a small degree of correlation (a condition we significantly extend in this paper that we will refer to as the full surplus extraction (FSE) condition), the seller can, in expectation, generate as much revenue as if she knew the bidders' true valuations. Further, correlated valuation is likely to be the norm, not the exception, in mechanism design settings because any valuation model with a common value component will be correlated.

However, the optimal mechanisms over correlated valuations are rarely, if ever, seen in practice due to the requirement that the mechanism designer knows precisely the prior distribution over bidders' values (Lopomo et al. 2009, Albert et al. 2015). If the mechanism designer tries to naïvely use an estimate of the distribution, the mechanism is unlikely to be incentive compatible or individually rational, leading to mechanisms that are hard to reason about and may perform very poorly. Therefore, if a mechanism designer intends to maximally exploit a correlated valuations setting, she must learn the distribution, and incorporate this information carefully into the mechanism design process.

As our primary contribution in this paper, we develop new computational techniques for a class of mechanisms, *robust mechanisms* (Bergemann and Morris 2005), that allow for some degree of uncertainty in the distribution over the bidders types while still performing better than weakly prior-dependent mechanisms such as ex-post mechanisms. We provide an algorithm for computing an optimal robust mechanism in polynomial time in the support of the distribution of types by combining techniques from the automated mechanism design literature (Conitzer and Sandholm 2002, 2004, Guo and Conitzer 2010, Sandholm and Likhodedov 2015) and the literature on robust optimization (Bertsimas and Sim 2004, Aghassi and Bertsimas 2006). We then introduce a novel class of mechanisms, *ϵ-robust mechanisms*, or mechanisms that guarantee with high probability that the standard constraints of incentive compatibility and individual rationality hold but allow for a non-zero chance of violation. We also show that we can construct an ϵ-robust mechanism with expected revenue that is an additive k-approximation to the revenue achievable by the optimal mechanism, for a setting in which full surplus extraction is possible, using a polynomial number of samples from the true distribution. Finally, we show experimentally that ϵ-robust mechanisms can significantly outperform other mechanism design procedures when the mechanism designer estimates the distribution over a bidder's type and a correlated external signal.

Additionally, we provide two additional contributions to the literature related to the problem of mechanism design in correlated valuation settings. First, we derive, for the first time, necessary and sufficient conditions for full surplus extraction as revenue for prior-dependent mechanisms under both ex-post and Bayesian incentive compatibility. These results extend the full surplus extraction as revenue condition in Cremer and McLean (1985, 1988) to a full characterization of the conditions necessary. We achieve this by considering the relationship between the valuation function and the information structure. By contrast, Cremer and McLean (Cremer and McLean 1988) characterize the conditions for full revenue extraction solely in terms of the information structure of the problem. Our conditions indicate that full surplus extraction as revenue is dependent on the rate of change in the bidder's belief in the value of an external signal. Moreover, we demonstrate a fundamental gap between mechanisms that are *ex-post incentive compatible* and those that are *Bayesian incentive compatible* by demonstrating that, in a correlated valuation setting, an ex-post incentive compatible mechanism guarantees, at most, a $(|\Theta| + 1)/4$ approximation to the revenue generated by a Bayesian incentive compatible mechanism, where $|\Theta|$ is the number of possible bidder types. These characterization results allow us to place strong approximation guarantees on ϵ-robust mechanisms, guarantees that would be impossible without the insight these results provide. However, we believe that these theoretical results may be of interest for more than guiding our design of robust mechanisms.

As our second additional contribution, we demonstrate the role the degree of correlation plays by demonstrating that for sufficiently low levels of correlation, learning the optimal mechanism is likely to be impossible. Specifically, we first consider the case of a countably infinite sequence of distributions, each satisfying the full surplus extraction condition, that converges to an independent private values distribution. We derive this negative result: no mechanism can guarantee revenue any higher than the optimal revenue for the IPV distribution over the entire sequence. Moreover, this remains true for any mechanism that has access to a finite number of samples from the underlying distribution. This implies that any mechanism that has access to a finite number of samples from the true distribution guarantees at most an approximation ratio of $(|\Theta|+1)/(2+\epsilon)$. This is in contrast to our positive results that an additive k-approximation to the optimal revenue is achievable with a polynomial number of samples. However, our positive result relies on a separation of beliefs that implies that the degree of correlation between bidders is not arbitrarily small, a condition that does not hold with probability one for a random distribution. Our negative result implies that instead of our separation of beliefs condition being a restriction on our mechanism design paradigm, *the effective application of mechanism design techniques to correlated valuation settings when the distribution is uncertain will fundamentally rely on this separation of belief.*

1.1. Related Work

In the sample complexity of mechanism design literature, the most closely related paper is Fu et al. (2014), which explores the sample complexity of optimal mechanism design with correlated valuations. They are able to show that if there is a finite set of distributions from which the true distribution will be drawn, then the sample complexity is of the same order as the number of possible distributions. However, the results are in a sense *too* strong. Specifically, their findings suggest that maximizing revenue from settings with correlated distributions with finite types is trivial from a sample complexity standpoint, at least if the set of possible distributions is known. Moreover, outside of a very small condition (effectively stating that there *is* correlation), the degree of correlation does not play a role in the ability to implement the mechanism, an intuitively strange result. The key to reconciling this intuition with their results is realizing that there is something fundamentally distinct between infinite sets of distributions and finite sets, and that their results do not extend to the case of infinite sets of distributions, as we demonstrate in Section 5. Moreover, in any setting of practical interest, the mechanism designer will face an infinite number of potential distributions.

In the traditional mechanism design literature, this work is closely related to work on *robust mechanism design* (Bergemann and Morris 2005, Lopomo et al. 2009). This line of literature assumes that the bidders in the mechanism have a belief over other bidders, but that the belief for each agent is not known by the mechanism designer, similar to our notion of uncertainty over the distribution. Instead, the belief of each bidder over other bidders becomes part of the "type" of the agent, following Harsanyi (1967). This is in some ways more flexible then our approach, where we define a bidder's type as his *payoff* type, while his belief is unknown but is from a known set. Since we are primarily interested in uncertain, but well-defined, beliefs, our notation will be sufficiently flexible for this work, and it allows us to more easily go beyond well defined beliefs. Our results differ from this previous work in two main respects. First, we develop and analyze a computational framework for computing a new class of mechanisms that satisfy the properties of Bergemann and Morris (2005) and Lopomo et al. (2009), whereas previous work is primarily interested in the theoretical limitations of such mechanism. Second, we extend beyond their definition of uncertainty in that we allow for probabilistic violation of the standard constraints in mechanism design. This leads to an entirely new class of mechanisms, of which we analyze the performance both theoretically and in simulation.

1.2. Outline of Paper

In Section 2, we define the problem setting for the perfect information case, and we provide notation that will be used throughout the paper. Then in Section 3, we derive necessary and sufficient conditions for full surplus extraction as revenue in a correlated valuation setting where the true distribution is known. Section 4 introduces our notion of uncertainty in the distribution. Next, in Section 5, we show the impossibility of learning the optimal mechanism using a finite number of samples from the underlying distribution for an arbitrary true distribution. This section demonstrates formally what is unrealistic about the full surplus extraction results (both those introduced in this work and in previous literature (Cremer and McLean 1985, 1988)). In Section 6, we introduce a computationally efficient methodology to deal with correlated valuation setting with robust mechanisms. This class of mechanisms is both computationally efficient, and it spans the space between ex-post and Bayesian mechanisms. We also relax the traditional constraints of strict individual rationality and incentive compatibility in order to introduce a computationally and sample efficient mechanism design procedure, based on ϵ-robust mechanisms, that allows for the probabilistic violation of the constraints. We then demonstrate that near optimal revenue can be achieved with a polynomial number of samples from the true distribution, and we demonstrate that, in simulation, this mechanism design procedure significantly outperforms traditional approaches. Section 7 concludes.

2. Preliminaries

We consider a single monopolistic seller auctioning one object, which the seller values at zero, to a single bidder whose valuation is correlated with an external signal. The special case of a single bidder and an externally verifiable signal captures many of the important aspects of this problem while increasing ease of exposition relative to the case of many bidders, and this setting has been used in the literature on correlated mechanism design (McAfee and Reny 1992, Albert et al. 2015) for this purpose. The external signal can, but does not necessarily, represent other bidders' bids.

The bidder has a *valuation* type θ drawn from a finite set of discrete types $\Theta = \{1, ..., |\Theta|\}$. Further, the bidder has a valuation function $v : \Theta \to \mathbb{R}^+$ that maps types to valuations for the object. Assume, without loss of generality, that for all $\theta, \theta' \in \Theta$, if $\theta > \theta'$ then $v(\theta) \geq v(\theta')$, and $v(1) > 0$. The discrete external signal is denoted by $\omega \in \Omega = \{1, 2, \ldots, |\Omega|\}$. Throughout the paper, we will denote vectors, matrices, and tensors as bold symbols.

There is a probability distribution, $\boldsymbol{\pi}$, over the types of the bidder and external signal where the probability of type and signal (θ, ω) is $\pi(\theta, \omega)$. The probability distribution can be represented in many possible ways, but we will represent it as a matrix. Specifically, the distribution is a matrix of dimension $|\Theta| \times |\Omega|$ whose elements are all positive and sum to one. Note that in contrast to much of the literature on mechanism design, we do not require that the bidder type be distributed independently of the external signal.

The distribution over the external signal ω given θ will be denoted by the $|\Omega|$ dimensional vector $\boldsymbol{\pi}(\cdot|\theta)$. We will, in many cases, be primarily interested in the conditional distribution over the external signal given the bidder's type, $\boldsymbol{\pi}(\cdot|\theta)$, so we will represent the full distribution as a marginal distribution over Θ, $\boldsymbol{\pi}_\theta$, and a set of conditional distributions over Ω, $\boldsymbol{\pi}(\cdot|\cdot) = \{\boldsymbol{\pi}(\cdot|1), \boldsymbol{\pi}(\cdot|2), ..., \boldsymbol{\pi}(\cdot||\Theta|)\}$. Therefore, if the true distribution is $\boldsymbol{\pi}$, we will alternatively represent it as $\{\boldsymbol{\pi}_\theta, \boldsymbol{\pi}(\cdot|\cdot)\}$. If for all $\theta, \theta' \in \Theta$, $\boldsymbol{\pi}(\cdot|\theta) = \boldsymbol{\pi}(\cdot|\theta')$, it as an independent private values (IPV) setting and the optimal mechanism is a *reserve price mechanism*, a mechanism where the seller makes a take it or leave it offer at the *reserve price* (Myerson 1981).

A (direct) revelation mechanism is defined by, given the bidder type and external signal (θ, ω), 1) a probability that the seller allocates the item to the bidder and 2) a monetary transfer from the bidder to the seller. We will denote the probability of allocating the item to the bidder as $p(\theta, \omega)$, which is a value between zero and one, and the transfer from the bidder to the seller as $x(\theta, \omega)$, where a positive value denotes a payment to the seller and a negative value a payment from the seller to the bidder. We will denote a mechanism as $(\boldsymbol{p}, \boldsymbol{x})$.

Definition 1 (Bidder's Utility). Given a realization of the external signal ω, reported type $\theta' \in \Theta$, and true type $\theta \in \Theta$, the bidder's utility under mechanism $(\boldsymbol{p}, \boldsymbol{x})$ is:

$$U(\theta, \theta', \omega) = v(\theta)p(\theta', \omega) - x(\theta', \omega)$$

Definition 2 (Bidder's Expected Utility). Given a reported type $\theta' \in \Theta$, true type $\theta \in \Theta$, and belief over the external signal $\boldsymbol{\pi}(\cdot|\theta)$, the bidder's expected utility under mechanism $(\boldsymbol{p}, \boldsymbol{x})$ is:

$$U(\theta, \theta') = \sum_{\omega \in \Omega} \pi(\omega|\theta)(v(\theta)p(\theta', \omega) - x(\theta', \omega))$$

Due to the well-known revelation principle (e.g., Gibbons (1992)), the seller can restrict her attention to incentive compatible mechanisms, i.e., mechanisms where it is always optimal for the bidder to truthfully report his valuation. However, incentive compatibility can be specified in multiple ways. For the sake of presentation, we will restrict our focus to two of the most common, *ex-post incentive compatibility* and *Bayesian incentive compatibility*. Ex-post incentive compatible mechanisms guarantee that for any realization of the external signal, the bidder always finds it optimal to report his value truthfully. In contrast, Bayesian incentive compatible mechanisms only guarantee that, given the beliefs of the bidder over the external signal, the bidder will have the highest *expected* utility if he reports truthfully: after seeing the realization of the external signal, he may regret his report.

Definition 3 (Ex-Post Incentive Compatibility). A mechanism $(\boldsymbol{p}, \boldsymbol{x})$ is *ex-post incentive compatible (IC)* if:

$$\forall \theta, \theta' \in \Theta, \omega \in \Omega : U(\theta, \theta, \omega) \geq U(\theta, \theta', \omega)$$

Definition 4 (Bayesian Incentive Compatibility). A mechanism $(\boldsymbol{p}, \boldsymbol{x})$ is *Bayesian incentive compatible (IC)* if:

$$\forall \theta, \theta' \in \Theta, U(\theta, \theta) \geq U(\theta, \theta')$$

Bayesian incentive compatibility is a statement about the *beliefs* of the bidder over the external signal, $\boldsymbol{\pi}(\omega|\theta)$. Specifically, it allows the seller to determine payments by *lottery*. The lottery that the bidder faces can be dependent on his valuation, but the lottery itself is over the external signal. Bayesian incentive compatibility is a strict relaxation of ex-post in the sense that any mechanism that is ex-post incentive compatible is also Bayesian incentive compatible.

In addition to incentive compatibility, we are interested in mechanisms that are *individually rational*, i.e., it is rational for a bidder to participate in the mechanism. We will define *ex-post individual rationality* (a bidder is *never* worse off by participating in the mechanism) and *ex-interim individual rationality* (the bidder has non-negative *expected* utility for participating in the mechanism). Again, ex-interim individual rationality is a strict relaxation of ex-post.

Definition 5 (Ex-Post Individual Rationality). A mechanism $(\boldsymbol{p}, \boldsymbol{x})$ is *ex-post individually rational (IR)* if:

$$\forall \theta \in \Theta, \omega \in \Omega : U(\theta, \theta, \omega) \geq 0$$

Definition 6 (Ex-Interim Individual Rationality). A mechanism $(\boldsymbol{p}, \boldsymbol{x})$ is *ex-interim individually rational (IR)* if:

$$\forall \theta \in \Theta : U(\theta, \theta) \geq 0$$

We will refer to mechanisms that satisfy ex-post individual rationality and incentive compatibility as *ex-post mechanisms* and mechanisms that satisfy Bayesian incentive compatibility and ex-interim individual rationality as *Bayesian mechanisms.* Bayesian mechanisms are what we have been referring to as prior-dependent mechanisms, while ex-post is weakly prior-dependent, i.e., only the objective function depends on the distribution, not the constraints over incentive compatibility and individual rationality.

Definition 7 (Optimal Mechanisms). A mechanism (q, m) is an *optimal ex-post mechanism* if under the constraint of ex-interim individual rationality and ex-post incentive compatibility it maximizes the following:

$$\sum_{\theta,\omega} x(\theta, \omega)\pi(\theta, \omega) \tag{1}$$

A mechanism that maximizes the above under the constraint of ex-interim individual rationality and Bayesian incentive compatibility is an *optimal Bayesian mechanism.*

The definition of an optimal mechanism combined with the constraints for individual rationality, incentive compatibility, and feasibility (i.e. that the item can be allocated at most once) define a linear optimization problem. An optimal mechanism can be efficiently computed as a solution to this linear program (Guo and Conitzer 2010).

Definition 8 (Full Social Surplus Extraction as Revenue). We say that a mechanism *extracts the full social surplus as revenue in expectation* if:

$$\sum_{\theta,\omega} \pi(\theta, \omega)x(\theta, \omega) = \sum_{\theta,\omega} \pi(\theta, \omega)v(\theta). \tag{2}$$

To illustrate the importance of prior-dependent mechanisms, it is necessary to review an important result in the literature on revenue maximization with correlated valuation distributions when the distribution is perfectly known.

Definition 9 (Cremer-McLean Condition). The distribution over types $\boldsymbol{\pi}$, is said to satisfy the Cremer-McLean condition if the set of beliefs associated with the bidder, $\{\boldsymbol{\pi}(\cdot|\theta) : \theta \in \Theta\}$, are linearly independent.

Theorem 1 **(Cremer and McLean (1985))**. *If the Cremer-McLean condition is satisfied by the distribution $\boldsymbol{\pi}$, then there exists an ex-interim IR and ex-post IC mechanism that extracts the full social surplus as revenue.*

This result due to Cremer and McLean (1985) states that under the generic Cremer-McLean condition (i.e. a condition that holds with probability one for a random distribution), the mechanism designer can generate as much revenue in expectation as if she knew the bidder's valuation. Therefore, with correlation, private information has no value for the bidder. This is in sharp contrast to the IPV setting where a consequence of the bidder having private information is that the seller must share some of the expected social surplus from the sell with the bidder. This is the strongest possible revenue guaranty in mechanism design.

We will reference the following in order to demonstrate a gap between optimal mechanisms, both in different classes and different assumptions about information over the distribution.

Example 1. Let the marginal distribution over the type of the bidder be given by $\pi(\theta) = 1/2^\theta$ for $\theta \in \{1, ..., |\Theta| - 1\}$ and $\pi(|\Theta|) = 1/2^{|\Theta|-1}$. Further let the value of the bidder for the item be $v(\theta) = 2^\theta$. Therefore, the expected value of the bidder's valuation is

$$\sum_{\theta=1}^{|\Theta|-1} \left(\frac{1}{2^\theta}\right) 2^\theta + \left(\frac{1}{2}\right)^{|\Theta|-1} 2^{|\Theta|} = |\Theta| + 1$$

Assume that the external signal is binary, i.e. $\Omega = \{\omega_L, \omega_H\}$. Note that for a reserve price mechanism with a reserve price of $2^{|\Theta|}$, the expected revenue is 2. Further, if the distribution is IPV, this is the optimal mechanism (Myerson 1981).

3. Necessary and Sufficient Conditions for Full Surplus Extraction as Revenue

In this section, we characterize necessary and sufficient conditions for full social surplus extraction as revenue given arbitrary correlation structures. Our results guarantee full surplus extraction even when $|\Omega| < |\Theta|$ or a subset of the conditional beliefs are a linear combination of others, both of which do not satisfy the Cremer-McLean Condition (Definition 9). We are able to extend Cremer and McLean (1985)'s result by considering the interaction between the prior $\boldsymbol{\pi}$ and the valuation function.

We will make use of the notion of a supergradient.

Definition 10 (Supergradient). Given some function $G : \mathbb{R}^{|\Omega|} \to \mathbb{R}$, a *supergradient to G at $\boldsymbol{f} \in \mathbb{R}^{|\Omega|}$* is a linear function $d : \mathbb{R}^{|\Omega|} \to \mathbb{R}$ such that $d(\mathbf{0}) = 0$ and for all $\boldsymbol{g} \in \mathbb{R}^{|\Omega|}$,

$$G(\boldsymbol{g}) \leq G(\boldsymbol{f}) + d(\boldsymbol{g} - \boldsymbol{f}). \tag{3}$$

We will make use of the following lemma.

LEMMA 1. *A mechanism $(\boldsymbol{p},\boldsymbol{x})$ extracts full surplus if and only if $p(\theta,\omega)=1$ and $U(\theta,\theta)=0$ for all $\theta\in\Theta,\omega\in\Omega$.*

The proof of Lemma 1 is straightforward.

THEOREM 2 **(Full Surplus Extraction with Ex-Post IC)**. *For a given $(\boldsymbol{\pi},V,\Omega)$, full surplus extraction is possible for an ex-post incentive compatible and ex-interim individually rational mechanism if and only if there exists a* linear *function $G:\mathbb{R}^{|\Omega|}\to\mathbb{R}$ such that $G(\boldsymbol{\pi}(\cdot|\theta))=v(\theta)$.*

Proof. First, assume that there exists an ex-post incentive compatible and ex-interim individually rational mechanism $(\boldsymbol{p},\boldsymbol{x})$ such that full surplus extraction is achieved in expectation. Then, by Definition 3 and Lemma 1, for all $\theta,\theta'\in\Theta$ and $\omega\in\Omega$:

$$v(\theta)-x(\theta,\omega)=v(\theta)-x(\theta',\omega).$$

Therefore for all $\theta,\theta'\in\Theta$ and $\omega\in\Omega$,

$$x(\theta,\omega)=x(\theta',\omega)=x^*(\omega).$$

Set, for $\boldsymbol{f}\in\mathbb{R}^{|\Omega|}$, $G(\boldsymbol{f})=\sum_{\omega\in\Omega}x^*(\omega)\boldsymbol{f}(\omega)$. Then, for $a,b\in\mathbb{R}$ and $\boldsymbol{f},\boldsymbol{g}\in\mathbb{R}^{|\Omega|}$,

$$\begin{aligned}G(a\boldsymbol{f}+b\boldsymbol{g})&=\sum_{\omega\in\Omega}x^*(\omega)(a\boldsymbol{f}(\omega)+b\boldsymbol{g}(\omega))\\&=a\sum_{\omega\in\Omega}x^*(\omega)\boldsymbol{f}(\omega)+b\sum_{\omega\in\Omega}x^*(\omega)\boldsymbol{g}(\omega)\\&=aG(\boldsymbol{f})+bG(\boldsymbol{g})\end{aligned}$$

Further, $G(\boldsymbol{\pi}(\cdot|\theta))=\sum_{\omega\in\Omega}x(\theta,\omega)\pi(\omega|\theta)=v(\theta)$ by ex-interim IR and Lemma 1.

Alternatively, suppose that there does exist a linear function $G:\mathbb{R}^{|\Omega|}\to\mathbb{R}$ such that $G(\boldsymbol{\pi}(\cdot|\theta))=v(\theta)$. Denote by $\boldsymbol{\pi}_i^*\in\Delta(\Omega)$ the probability distribution such that $\boldsymbol{\pi}_i^*(\omega=i)=1$ and $\boldsymbol{\pi}_i^*(\omega\neq i)=0$. For all $\theta\in\Theta$ and $\omega\in\Omega$, set $p(\theta,\omega)=1$ and $x(\theta,\omega)=G(\boldsymbol{\pi}_\omega^*)$. Then,

$$\begin{aligned}\sum_{\omega\in\Omega}x(\theta,\omega)\boldsymbol{\pi}(\omega|\theta)&=\sum_{\omega\in\Omega}G(\boldsymbol{\pi}_\omega^*)\boldsymbol{\pi}(\omega|\theta)\\&=G\left(\sum_{\omega\in\Omega}\boldsymbol{\pi}_\omega^*\boldsymbol{\pi}(\omega|\theta)\right)\\&=G(\boldsymbol{\pi}(\cdot|\theta))=v(\theta).\end{aligned}$$

Therefore, ex-interim IR is satisfied and binding, and given that for all $\theta,\theta'\in\Theta$, $x(\theta,\omega)=x(\theta',\omega)=x^*(\omega)=G(\boldsymbol{\pi}_i^*)$, ex-post IC is satisfied. □

Intuitively, Theorem 2 states that ex-post IC combined with ex-interim IR allow for the mechanism designer to incorporate only a single lottery over the external signal into the ex-post mechanism. This lottery is such that the payoff for $\omega = i$ is the linear function G evaluated at $\boldsymbol{\pi}_i^*$ (defined as in the above proof). Full surplus extraction is only possible, then, when one lottery, or linear function, can intersect every valuation. By contrast, the additional power of a Bayesian mechanism is that the mechanism designer can incorporate many lotteries.

THEOREM 3 **(Full Surplus Extraction with Bayesian IC)**. *For a given $(\boldsymbol{\pi}, V, \Omega)$, full surplus extraction is possible for a Bayesian incentive compatible, ex-interim individually rational mechanism if and only if there exists a* concave *function $G : \mathbb{R}^{|\Omega|} \to \mathbb{R}$ such that $G(\boldsymbol{\pi}(\cdot|\theta)) = v(\theta)$.*

Proof. Assume that there exists a Bayesian incentive compatible, ex-interim individually rational mechanism $(\boldsymbol{p}, \boldsymbol{x})$ such that the full surplus is extracted in expectation. Suppose in addition that there does not exist a concave function $G : \mathbb{R}^{|\Omega|} \to \mathbb{R}$ such that $G(\boldsymbol{\pi}(\cdot|\theta)) = v(\theta)$. This implies, by the definition of concavity, that for every function $G^* : \mathbb{R}^{|\Omega|} \to \mathbb{R}$ such that $G^*(\boldsymbol{\pi}(\cdot|\theta)) = v(\theta)$ there must exist $\theta^* \in \Theta$ such that for $\theta \in \Theta \backslash \{\theta^*\}$, there exists $\alpha_\theta \geq 0$ where $\sum_{\theta \in \Theta \backslash \{\theta^*\}} \alpha_\theta = 1$, $\boldsymbol{\pi}(\omega|\theta^*) = \sum_{\theta \in \Theta \backslash \{\theta^*\}} \alpha_\theta \boldsymbol{\pi}(\omega|\theta)$, and

$$\sum_{\theta \in \Theta \backslash \{\theta^*\}} \alpha_\theta v(\theta) = \sum_{\theta \in \Theta \backslash \{\theta^*\}} \alpha_\theta G^*(\boldsymbol{\pi}(\cdot|\theta)) > G^* \left(\sum_{\theta \in \Theta \backslash \{\theta^*\}} \alpha_\theta \boldsymbol{\pi}(\omega|\theta) \right) = G^*(\boldsymbol{\pi}(\cdot|\theta^*)) = v(\theta^*).$$

Note that for all $\theta \in \Theta$, $U(\theta, \theta) = 0$ by Lemma 1. Then,

$$\begin{aligned} \sum_{\theta \in \Theta \backslash \{\theta^*\}} \alpha_\theta U(\theta, \theta^*) &= \sum_{\theta \in \Theta \backslash \{\theta^*\}} \alpha_\theta \left(\sum_\omega \pi(\omega|\theta)(v(\theta) - x(\theta^*, \omega)) \right) \\ &> v(\theta^*) - \sum_\omega x(\theta^*, \omega) \sum_{\theta \in \Theta \backslash \{\theta^*\}} \alpha_\theta \pi(\omega|\theta) \\ &= v(\theta^*) - \sum_\omega x(\theta^*, \omega) \pi(\omega|\theta^*) \\ &= v(\theta^*) - v(\theta^*) = 0 \end{aligned}$$

Then, there exists $\theta' \in \Theta \backslash \{\theta^*\}$ such that $U(\theta', \theta^*) > 0 = U(\theta', \theta')$, in contradiction of Bayesian IC. Therefore, there must exist a concave function $G : \mathbb{R}^{|\Omega|} \to \mathbb{R}$ such that $G(\boldsymbol{\pi}(\cdot|\theta)) = v(\theta)$.

Now, assume that there does exist a concave function $G : \mathbb{R}^{|\Omega|} \to \mathbb{R}$ such that $G(\boldsymbol{\pi}(\cdot|\theta)) = v(\theta)$. Let d_θ be a supergradient to G at $\boldsymbol{\pi}(\cdot|\theta)$. This exists everywhere by the concavity of G. Denote by $\boldsymbol{\pi}_i^* \in \Delta(\Omega)$ the probability distribution such that $\pi_i^*(\omega = i) = 1$ and $\pi_i^*(\omega \neq i) = 0$. Then, for all $\theta \in \Theta$ and $\omega \in \Omega$ set $p(\theta, \omega) = 1$, and the transfers such that

$$x(\theta, \omega) = G(\boldsymbol{\pi}(\cdot|\theta)) + d_\theta(\boldsymbol{\pi}_\omega^* - \boldsymbol{\pi}(\cdot|\theta)).$$

Then

$$\begin{aligned}U(\theta,\theta) &= \sum_{\omega}(v(\theta)-x(\theta,\omega))\pi(\omega|\theta)\\ &= v(\theta)-\sum_{\omega}(G(\boldsymbol{\pi}(\cdot|\theta))+d_\theta(\boldsymbol{\pi}^*_\omega-\boldsymbol{\pi}(\cdot|\theta)))\pi(\omega|\theta)\\ &= v(\theta)-G(\boldsymbol{\pi}(\cdot|\theta))-d_\theta\left(\sum_{\omega}\boldsymbol{\pi}^*_\omega\pi(\omega|\theta)-\boldsymbol{\pi}(\cdot|\theta)\right)\\ &= v(\theta)-v(\theta)-d(0)=0.\end{aligned}$$

Therefore, ex-interim IR binds for all $\theta\in\Theta$. Also, for all $\theta,\theta'\in\Theta$

$$\begin{aligned}U(\theta,\theta') &= \sum_{\omega}(v(\theta)-x(\theta',\omega))\pi(\omega|\theta)\\ &= v(\theta)-\sum_{\omega}(G(\boldsymbol{\pi}(\cdot|\theta'))+d_\theta(\boldsymbol{\pi}^*_\omega-\boldsymbol{\pi}(\cdot|\theta')))\pi(\omega|\theta)\\ &= v(\theta)-G(\boldsymbol{\pi}(\cdot|\theta'))-d_\theta(\sum_{\omega}\boldsymbol{\pi}^*_\omega\pi(\omega|\theta)-\boldsymbol{\pi}(\cdot|\theta'))\\ &= G(\boldsymbol{\pi}(\cdot|\theta))-G(\boldsymbol{\pi}(\cdot|\theta'))-d_\theta(\boldsymbol{\pi}(\cdot|\theta)-\boldsymbol{\pi}(\cdot|\theta'))\\ &\le 0=U(\theta,\theta)\end{aligned}$$

Therefore, by Lemma 1, the mechanism extracts full surplus. □

Theorem 3 is able to relax the necessity of a linear function G by using multiple lotteries. Each lottery corresponds to a linear function, just as in Theorem 2, but the linear function is a supergradient of a concave function. The concavity of the function ensures that each bidder finds it IC to only participate in the lottery that corresponds to his type. Figure 1c depicts this graphically for $|\Theta|=3$ and $|\Omega|=2$. Note that this result is reminiscent of *proper scoring rules* (Gneiting and Raftery 2007, Frongillo and Kash 2014). Corollary 1 shows the connection between our full surplus extraction condition and the Cremer-McLean condition.

Corollary 1. *If the Cremer-McLean Condition (Definition 9) holds, then there exists a linear function $G:\mathbb{R}^{|\Omega|}\to\mathbb{R}$ such that $G(\boldsymbol{\pi}(\cdot|\theta))=v(\theta)$.*

Proof. Let the set of beliefs $\{\boldsymbol{\pi}(\cdot|\theta):\theta\in\Theta\}$ satisfy the Cremer-McLean condition (Definition 9), and let $\boldsymbol{v}\in\mathbb{R}^{|\Theta|}$ be such that $\boldsymbol{v}(\theta)=v(\theta)$. Define a matrix $\Gamma\in\mathbb{R}^{|\Theta|}\times\mathbb{R}^{|\Omega|}$ such that

$$\boldsymbol{\Gamma}=\begin{bmatrix}\boldsymbol{\pi}(1|1) & \cdots & \boldsymbol{\pi}(|\Omega||1)\\ \vdots & \ddots & \vdots\\ \boldsymbol{\pi}(1||\Theta|) & \cdots & \boldsymbol{\pi}(|\Omega|||\Theta|)\end{bmatrix} \tag{4}$$

Then by the assumption of linear independence, Γ is full rank, and there exists $\boldsymbol{c}\in\mathbb{R}^{|\Omega|}$ such that $\Gamma\boldsymbol{c}=\boldsymbol{v}$.

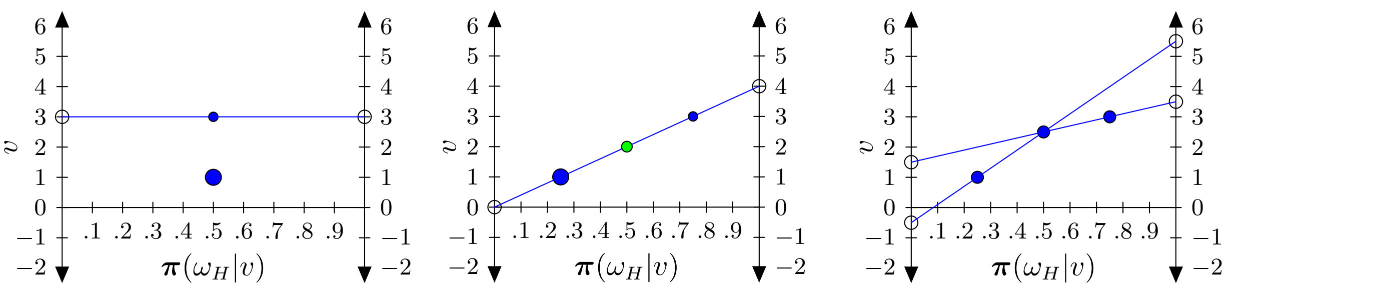
Define $G:\mathbb{R}^{|\Omega|}\to\mathbb{R}$ such that for $\boldsymbol{f}\in\mathbb{R}^{|\Omega|}$, $G(\boldsymbol{f})=\sum_{\omega\in\Omega}c(\omega)f(\omega)$. Then, G is linear and $G(\boldsymbol{\pi}(\cdot|\theta))=v(\theta)$. □

Figure 1 The points represent the bidder type, where the position along the x-axis is the probability that the external signal is high. The relative size of the point represents the marginal probability of that bidder type. The lines represent lotteries offered in the mechanism, with the payment for the lottery if ω_H is observed being the intersection with the right vertical axis, and the payment if ω_L is observed is the intersection with the left axis. The height of the line at each point is the expected payment for that lottery. The bidder accepts a lottery if and only if the expected payment is less than or equal to his valuation (IR) and chooses the lottery with the lowest expected payment (IC). For these mechanisms, if a bidder accepts a lottery, the item is allocated with probability 1. Figure 1a shows a take it or leave it offer of 3, and only the high valuation $v = 3$ is allocated the item.

(a) Distribution where the valuation is uncorrelated with the external signal, $\boldsymbol{\pi}_1$

(b) Distribution that satisfies the Cremer-McLean Condition, $\boldsymbol{\pi}_2$ (blue dots), and one that does not (blue and green dots), $\boldsymbol{\pi}_3$, but satisfies Theorem 2.

(c) Distribution that satisfies the FSE condition but fails Cremer-Mclean, $\boldsymbol{\pi}_3$ with $v \in \{1, 2.5, 3\}$.



We will refer to any distribution $\boldsymbol{\pi}$ that satisfies the conditions for full surplus extraction as in Theorem 3 as satisfying the *full surplus extraction* (FSE) condition. Example 2 and Figure 1 demonstrates the full surplus extraction conditions, and gives example mechanisms that extract full surplus as revenue.

Example 2. Suppose that there is a single bidder and an external signal that is correlated with the bidder's valuation. Both the bidder valuations and the external signal are binary, and we will denote the bidder valuations by $v \in \{1, 3\}$ and the possible values of the external signal by $\omega \in \{\omega_L, \omega_H\}$. Denote the distribution of the bidder's valuations and the external signal by

$$\boldsymbol{\pi}_1(v,\omega) = \begin{bmatrix} 1/3 & 1/3 \\ 1/6 & 1/6 \end{bmatrix} \qquad \boldsymbol{\pi}_2(v,\omega) = \begin{bmatrix} 1/2 & 1/6 \\ 1/12 & 1/4 \end{bmatrix}$$

where the indices are ordered such that $\boldsymbol{\pi}_2(v = 3, \omega = \omega_L) = 1/12$. Note that the marginal distributions over v are identical for $\boldsymbol{\pi}_1$ and $\boldsymbol{\pi}_2$. It is clear that in $\boldsymbol{\pi}_1$ the bidder's valuation and the external signal are uncorrelated, implying that the optimal mechanism is a reserve price mechanism (Myerson 1981), shown in Figure 1a, with an expected revenue of 1. However, $\boldsymbol{\pi}_2$ satisfies the Cremer-McLean condition, and therefore, the seller can extract full surplus as revenue (the full 5/3), as in Figure 1b (blue dots only).

Now suppose that the set of valuations is instead $v \in \{1,2,3\}$, and the distribution over types and external signals is given by

$$\boldsymbol{\pi}_3(v,\omega) = \begin{bmatrix} 1/4 & 1/12 \\ 1/6 & 1/6 \\ 1/12 & 1/4 \end{bmatrix}$$

Given that $|\Theta| > |\Omega|$, $\boldsymbol{\pi}_3$ does not satisfy the Cremer-McLean Condition. However, it does satisfy conditions for Theorem 2, and Figure 1b (blue and green dots) shows that the same mechanism that extract full surplus for the setting that satisfies the Cremer-McLean condition still extracts full surplus.

Finally, if the set of valuations is $v \in \{1,2.5,3\}$, and if the distribution over types and external signals is again given by $\boldsymbol{\pi}_3$ (depicted in Figure 1c), then it is trivial to verify that the distribution does not satisfy either the Cremer-McLean condition or the condition for Theorem 2. However, it does satisfy the FSE condition, and Figure 1c demonstrates a mechanism that extracts full surplus.

THEOREM 4. *The expected revenue generated by an optimal ex-post incentive compatible and ex-interim individually rational mechanism guarantees at most a* $(|\Theta|+1)/4$ *approximation to the expected revenue generated by an optimal Bayesian incentive compatible and ex-interim individually rational mechanism.*

The proof is in the Appendix.

3.1. Simulation Results for Optimal Revenue Under Full Information

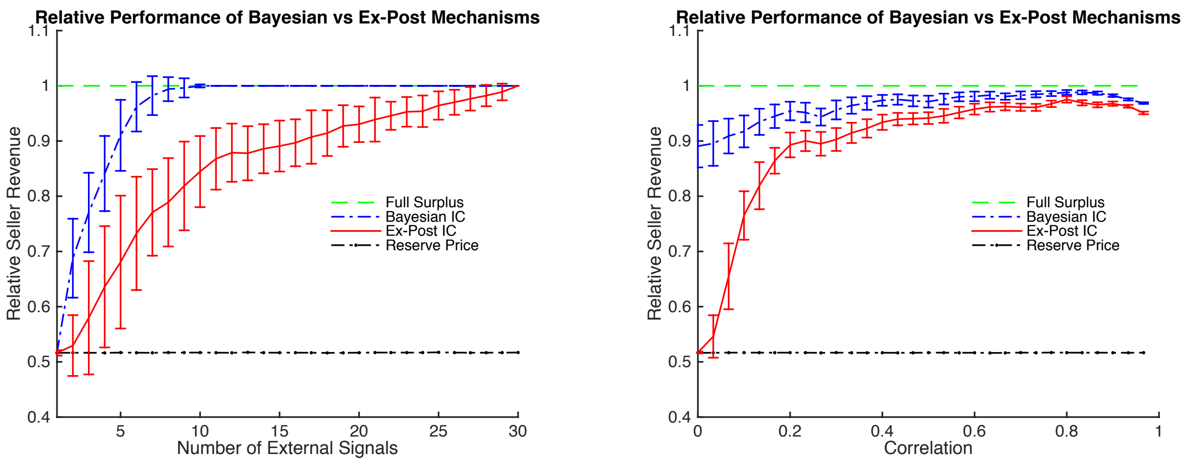
To explore the relative importance of Bayesian versus ex-post IC for revenue efficiency, we generate random distributions and solve for the optimal mechanism using automated mechanism design techniques (Conitzer and Sandholm 2002, Albert et al. 2015) under both instances of Bayesian and ex-post IC. We construct each distribution by generating 1500 samples, where a sample consists of two independent draws from the uniform distribution over $[0,1]$ corresponding to a realization of θ and ω. Denote sample i by $\boldsymbol{x}_i = (\theta_i, \omega_i)$. We pick a target correlation ρ, and construct the 2×2 correlation matrix C. Then, we decompose the correlation matrix using the Cholesky decomposition to calculate L such that $C = LL^T$. Finally, we transform our independent values for θ and ω into correlated values by constructing the final sample as $\boldsymbol{x}'_i = \boldsymbol{x}_i L^T$. This guarantees that $corr(\boldsymbol{x}'(1), \boldsymbol{x}'(2)) = \rho$ within the sample.

We then construct a discrete probability distribution from the 1500 samples by calculating equally spaced buckets along $[0,1]^2$, where the number of buckets along each dimension is equal to the number of bidder types and number of external signals. Then we count how many samples fall in each bucket and normalize. The valuations corresponding to each sample is the upper limit of the bucket that contains the sample. Note that due to the way that we correlate the two independent variables, there are some samples where $\omega_i > 1$. In that case, we count them in the highest bucket.

Figure 2 Relative performance of the optimal Bayesian vs Ex-post IC mechanisms for randomly generated distributions. $|\Theta| = 30$, $|\Omega| = 5$, and $\rho = .1$ unless otherwise specified. Each point is the average of 100 randomly generated distributions. 95% confidence intervals shown. All mechanisms are ex-interim IR.

(a) The effect of varying the external signal on optimal revenue. We vary $|\Omega|$ from $\{1, ..., 30\}$.

(b) The effect of varying correlation on optimal revenue. We vary ρ from $[0, 1]$ with a step size of $1/30$.



This procedure allows us to generate correlated joint distributions randomly with each distribution unique, as well as allowing us to independently vary both the number of external signals and the degree of correlation. Note that with probability one, if the number of external signals $|\Omega|$ is equal to the number of valuations $|\Theta|$, then the Cremer-McLean condition (Definition 9) will be satisfied and full social surplus extraction will be possible using an ex-post IC mechanism. This guarantees that as we increase the number of external signals, we should converge to full social surplus extraction. However, as can be seen in Figure 2a, the optimal Bayesian mechanism converges to full extraction with much fewer external signals than the optimal ex-post mechanism, becoming indistinguishable from full surplus extraction with $|\Omega| = 10$, while the optimal ex-post IC mechanism generates significantly less revenue until the number of external states equals the number of bidder types.

In Figure 2b, we vary the degree of correlation, while holding the number of external states constant. We observe that as the correlation between the bidder's type and the external signal approaches 1, both the optimal Bayesian and ex-post mechanisms get very close to full revenue extraction. However, the optimal Bayesian mechanism generates a statistically significant larger amount of revenue than the optimal ex-post mechanism for all correlation values.

Given Theorem 4 and the results displayed in Figure 2, we will exclusively focus on Bayesian mechanisms throughout the remainder of this work.

4. Consistent Sets of Distributions

While Theorems 1, 2, and 3 make relatively weak assumptions about the distributions in order to guarantee full revenue extraction, they do require that the mechanism designer knows the distribution exactly. If, instead of precise knowledge of the distribution of bidder types and external signals, the mechanism designer has an imprecise estimate of the distribution, the prior-dependent, or Bayesian, mechanism can fail to be both incentive compatible and individually rational. This failure can be a significant problem for two reasons. First, if the mechanism is not individually rational bidders will not participate in the mechanism. If the market is thin, the loss of even a single bidder can lead to significant decreases in expected revenue, even relative to simple mechanisms (Bulow and Klemperer 1996). Second, if the mechanism is not incentive compatible, the bidder may optimally choose to mis-report his true valuation, leading both to biases in future estimates of the distribution and difficulty in reasoning about the performance of the mechanism, since it is unclear a priori how the bidder will report.

It is in this sense that Bayesian incentive compatible and ex-interim individually rational mechanisms are, in general, strongly prior-dependent. The mechanism depends not only on the seller's estimate of the distribution, but also the bidder's belief over the distribution. The consequences of these being mis-aligned is not just slightly lower expected revenue, as would be the case for weakly prior-dependent mechanisms such as a second price auction with reserve; it is a failure of the mechanism to maintain fundamental characteristics (Hartline 2014, Albert et al. 2015). Therefore, unless the seller has perfect knowledge of the bidder's beliefs, standard mechanism design techniques will leave only the option of using sub-optimal, weakly prior-dependent mechanisms.

A more realistic assumption is that the distribution is not perfectly known, but instead estimated, i.e. the seller estimates the distribution $\boldsymbol{\pi}$ as $\hat{\boldsymbol{\pi}}$. If this estimation is imperfect, then it is likely that there exists a set of distributions that are consistent with the estimated distribution.

Definition 11 (Set of Consistent Distributions). Let $P(A)$ be the set of probability distributions over a set A. Then the space of all probability distributions over $\Theta \times \Omega$ can be represented as the Cartesian product $P(\Theta) \times \prod_{\theta \in \Theta} P(\Omega)$. A subset $\mathcal{P}(\hat{\boldsymbol{\pi}}) = \mathcal{P}(\{\hat{\boldsymbol{\pi}}_\theta, \hat{\boldsymbol{\pi}}(\cdot|\cdot)\}) \subseteq P(\Theta) \times \prod_{\theta \in \Theta} P(\Omega)$ is a *consistent set of distributions* for the estimated distribution $\hat{\boldsymbol{\pi}} = \{\hat{\boldsymbol{\pi}}_\theta, \hat{\boldsymbol{\pi}}(\cdot|\cdot)\}$ if the true distribution, $\boldsymbol{\pi} = \{\boldsymbol{\pi}_\theta, \boldsymbol{\pi}(\cdot|\cdot)\}$, is guaranteed to be in $\mathcal{P}(\hat{\boldsymbol{\pi}})$.

While a consistent set is a set of joint distributions over both Θ and Ω, we will find it useful throughout the rest of the paper to refer to the set of consistent conditional distributions for $\theta \in \Theta$ as $\mathcal{P}(\hat{\boldsymbol{\pi}}(\cdot|\theta))$. Namely, in Bayesian IC and ex-interim IR mechanisms, it will be the conditional distributions that are essential. Similarly, the set of consistent marginal distributions over Θ will be referred to as $\mathcal{P}(\hat{\boldsymbol{\pi}}_\theta)$.

With a consistent set of distributions, we can relax the notion of ex-interim IR and Bayesian IC by requiring that the mechanism be IR and IC for all distributions in the consistent set. This is similar to the notion of bidder beliefs being part of the type space introduced in Bergemann and Morris (2005). In contrast to the previous work, we explicitly keep the uncertainty in the distribution separate from concerns over uncertainty in the type. However, since the distribution $\boldsymbol{\pi}$ is also private information, by the revelation principle, the mechanism designer can also elicit the true distribution from the bidder and condition the mechanism on the reported distribution. Therefore, we modify the definitions of the mechanism, $(\boldsymbol{p}, \boldsymbol{x})$, such that they depend not only on the reported type and external signal, but also the reported distribution $\boldsymbol{\pi}'$. We similarly modify the definition of bidder utility to depend not only on the reported type θ and external signal ω, but also on the reported, $\boldsymbol{\pi'}$, and true distribution, $\boldsymbol{\pi}$.

Definition 12 (Robust Individual Rationality). A mechanism is *robust individually rational* for estimated bidder distribution $\hat{\boldsymbol{\pi}}$ and consistent set of distributions $\mathcal{P}(\hat{\boldsymbol{\pi}})$ if for all $\theta \in \Theta$, and $\boldsymbol{\pi} \in \mathcal{P}(\hat{\boldsymbol{\pi}})$,

$$\sum_{\omega \in \Omega} \pi(\omega|\theta) U(\theta, \boldsymbol{\pi}, \theta, \boldsymbol{\pi}, \omega) \geq 0$$

Definition 13 (Robust Incentive Compatibility). A mechanism is *robust incentive compatible* for estimated bidder distribution $\hat{\boldsymbol{\pi}}$ and consistent set of distributions $\mathcal{P}(\hat{\boldsymbol{\pi}})$ if for all $\theta, \theta' \in \Theta$ and $\boldsymbol{\pi}, \boldsymbol{\pi}' \in \mathcal{P}(\hat{\boldsymbol{\pi}})$,

$$\sum_{\omega \in \Omega} \pi(\omega|\theta) U(\theta, \boldsymbol{\pi}, \theta, \boldsymbol{\pi}, \omega) \geq \sum_{\omega \in \Omega} \pi(\omega|\theta) U(\theta, \boldsymbol{\pi}, \theta', \boldsymbol{\pi}', \omega)$$

Note that we can restrict our attention to settings where the bidder only reports $\boldsymbol{\pi}' \in \mathcal{P}(\hat{\boldsymbol{\pi}})$ by setting the allocation probability $\boldsymbol{p}$ to zero if the bidder reports $\boldsymbol{\pi}' \notin \mathcal{P}(\hat{\boldsymbol{\pi}})$.

5. Converging Sequences of Distributions

Given that full surplus extraction as revenue is easy in the case of full knowledge of the distribution (in the sense that the necessary condition is generic), it is reasonable to ask whether it is easy to generate nearly full revenue by using samples from the underlying distribution. I.e., suppose that we start with some consistent set $\mathcal{P}(\hat{\boldsymbol{\pi}})$, and we have access to a finite number of samples from the true distribution $\boldsymbol{\pi}$, can we guarantee nearly optimal revenue for the true distribution by using the samples in the mechanism design process? In the setting where $\mathcal{P}(\hat{\boldsymbol{\pi}})$ is finite, the answer is yes. Fu et al. (2014) showed that if the set of possible distributions is finite, then with relatively few samples, full surplus extraction is possible. However, it is likely that any reasonable distribution estimation procedure will return a continuous and closed set of distributions that are consistent with the observed samples. Specifically, any non-parametric approach to the estimation will necessarily include an uncountably infinite set of possible distributions. In this section, we explore the possibility of computing the optimal mechanism with a finite number of samples, and we demonstrate that in the worst case, there is no finite number of samples that can approximate the full revenue mechanisms.

We will prove this for an even more restrictive set, that of countably infinite sequences of distributions that converge to a distribution. This set is more restrictive in the sense that any continuous closed set of distributions will contain an infinite number of these sequences. Since we will show that this more restrictive set is sufficient, the results will naturally extend to the more permissive set. We will further simplify this setting by assuming that we know the marginal distribution, $\boldsymbol{\pi}_\theta$, perfectly, and we must only estimate the conditional distributions $\{\boldsymbol{\pi}(\cdot|\theta)\}_{\theta\in\Theta}$. Again, uncertainty in the marginal distribution strictly increases the complexity of the problem, so our negative results will extend to the more general setting.

Definition 14 (Converging Distributions). A countably infinite sequence of distributions $\{\boldsymbol{\pi}_i\}_{i=1}^{\infty}$ is said to be *converging to the distribution* $\boldsymbol{\pi}^*$, the *convergence point*, if for all $\theta \in \Theta$ and $\epsilon > 0$, there exists a $T \in \mathbb{N}$ such that for all $i \geq T$, $||\boldsymbol{\pi}_i(\cdot|\theta) - \boldsymbol{\pi}^*(\cdot|\theta)|| < \epsilon$. I.e., for each $\theta \in \Theta$, the conditional distributions in the sequence, $\{\boldsymbol{\pi}_i(\cdot|\theta)\}_{i=1}^{\infty}$, converge to the conditional distribution $\boldsymbol{\pi}^*(\cdot|\theta)$ in the l^2 norm.

Note that in Definition 14, we do not explicitly assume that the elements of the sequence satisfy the FSE condition, nor do we assume that the distribution to which the sequence is converging is an IPV distribution. However, it is straightforward to construct examples of converging sequences such that every element of the sequence satisfies FSE but the limit is IPV. Figure 3a demonstrates one such set. We will make use of the following standard definition.

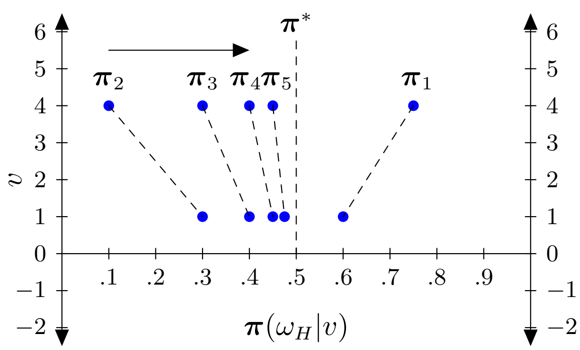
Definition 15 (Affine Independence). A set of vectors $\{\boldsymbol{v}_i\}_{i=1}^m$ over $\mathbb{R}^n$ are affinely independent if for $\{\alpha_i\}_{i=1}^m$, $\sum_i \alpha_i \boldsymbol{v}_i = \mathbf{0}$ and $\sum_i \alpha_i = 0$ implies $\alpha_i = 0$ for all $i \in \{1, ..., m\}$.

The set of distributions over Ω are the points on a $|\Omega|$-simplex where the vertices of the simplex are denoted by the set of distributions such that $\pi(\omega) = 1$ for all $\omega \in \Omega$ (see Figure 3b). Further, any set of distributions over Ω of size $|\Omega|$ that are affinely independent must span the $|\Omega|$-simplex with affine combinations. I.e., if the set $\{\boldsymbol{\pi}_i\}_{i=1}^{|\Omega|}$ is affinely independent, then for any distribution $\boldsymbol{\pi}'$ over Ω, there must exist $\{\alpha_i\}_{i=1}^{|\Omega|}$ where $\sum_i \alpha_i = 1$ and $\boldsymbol{\pi}' = \sum_i \alpha_i \boldsymbol{\pi}_i$.

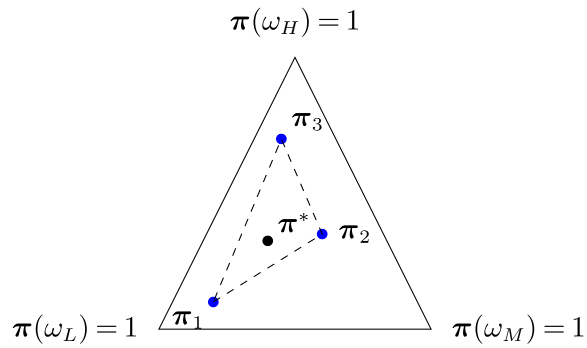
We can assume, without loss of generality, that for any sequence of distributions we consider, $\{\boldsymbol{\pi}_i\}_{i=1}^{\infty}$, there must exist a subset of $\{\boldsymbol{\pi}_i(\cdot|\theta)\}_{i,\theta}$ of size $|\Omega|$ that is affinely independent. If not, the affine combination of vectors $\{\boldsymbol{\pi}_i(\cdot|\theta)\}_{i,\theta}$ spans a lower dimensional simplex, and we can reduce the dimensionality of Ω until an affinely independent subset exists. Note that this relies on the

Figure 3 Figure 3a demonstrates a converging sequence as in Definition 14. Each point represents a conditional distribution, and conditional distributions linked by a dashed line both belong to the same full distribution. Specifically, the conditional distributions are all converging to $\boldsymbol{\pi}(\omega_H|v) = 1/2$, i.e. a distribution where the bidder's value and the external signal are uncorrelated. However, all of the distributions in the sequence satisfy the Cremer-McLean condition. Figure 3b demonstrates distributions that satisfy Assumption 1 in a set of distributions over three possible external signals $\{\omega_L, \omega_M, \omega_H\}$ as points in a 2-simplex. Specifically, $\boldsymbol{\pi}^*$ is a strictly convex combination of $\boldsymbol{\pi}_1$, $\boldsymbol{\pi}_2$, and $\boldsymbol{\pi}_3$. The sequence of distributions in Figure 3a also satisfies Assumption 1 due to $\boldsymbol{\pi}_1$, but if $\boldsymbol{\pi}_1$ was excluded from the sequence, it would not.

(a) Sequence of converging distributions with a binary signal.



(b) Sequence elements that satisfy Assumption 1.



assumption that the bidder is risk neutral. Specifically, a risk neutral bidder is indifferent between a payment for an outcome of the external signal, $p(\theta', \boldsymbol{\pi}_{i'}, \omega)$, and a lottery over multiple values of the external signal with the same expected payoff. Therefore, if there is not a subset of $\{\boldsymbol{\pi}_i(\cdot|\theta)\}_{i,\theta}$ of size $|\Omega|$ that is affinely independent we can always replace the true signal with a lower dimensional set of lotteries over the external signal without affecting the expected utility of the bidder.

In addition to Definition 14, we will require the following assumption.

Assumption 1. *For the sequence of distributions $\{\boldsymbol{\pi}_i\}_{i=1}^{\infty}$ converging to $\boldsymbol{\pi}^*$ and for any $\theta' \in \Theta$, there exists a subset of distributions of size $|\Omega|$ from the set $\{\boldsymbol{\pi}_i(\cdot|\theta)\}_{i,\theta}$ that is affinely independent and the distribution $\boldsymbol{\pi}^*(\cdot|\theta')$ is a strictly convex combination of the elements of the subset. I.e., there exists $\{\alpha_k\}_{k=1}^{|\Omega|}$, $\alpha_k \in (0,1)$ and $\{\boldsymbol{\pi}_k(\cdot|\theta_k)\}_{k=1}^{|\Omega|}$, where every $\boldsymbol{\pi}_k(\cdot|\theta_k) \in \{\boldsymbol{\pi}_i(\cdot|\theta)\}_{i,\theta}$, such that $\boldsymbol{\pi}^*(\cdot|\theta') = \sum_{k=1}^{|\Omega|} \alpha_k \boldsymbol{\pi}_k(\cdot|\theta_k)$.*

Assumption 1 states that the sequence of distributions is converging to a distribution that is in the interior of the sequence. This is not without loss of generality, but it is only violated for a measure zero set of distributions, the distributions at the boundary of the space of the strictly convex combinations of the distributions within the sequence of distributions. Specifically, Assumption 1 is a statement about the *conditional distributions*, and particularly that all conditional distributions of the convergence point, $\boldsymbol{\pi}^*(\cdot|\theta)$ for all θ, is in some sense in the interior of some other estimate (see

Figures 3b for a graphical depiction of this statement). Moreover, the distributions that "enclose" the convergence point do not have to have the same θ, i.e. any conditional distributions for any θ in the set $\{\boldsymbol{\pi}_i(\cdot|\theta)\}_{i,\theta}$ can be the distributions that "enclose" the convergence point. Further, if the full set of all potential distributions is a continuous closed set, then there will be an infinite number of sequences that satisfy this assumption.

With these definitions, we are able to introduce our main impossibility results, Theorem 5 and Corollary 2.

THEOREM 5. *Let $\{\boldsymbol{\pi}_i\}_{i=1}^{\infty}$ be a sequence of distributions converging to $\boldsymbol{\pi}^*$ that satisfies Assumption 1. Denote the revenue of the optimal mechanism for the distribution $\boldsymbol{\pi}^*$ by R. For any $k>0$, and for any mechanism that is incentive compatible and individually rational, there exists a $T\in\mathbb{N}$ such that for all $\boldsymbol{\pi}_{i'}\in\{\boldsymbol{\pi}_i\}_{i=T}^{\infty}$, the expected revenue is less than $R+k$.*

Theorem 5, whose proof we shall defer to the end of this section, states that *no mechanism* can guarantee revenue better than the optimal revenue achievable at the convergence point for all distributions in the sequence. Namely, if the sequence of distributions $\{\boldsymbol{\pi}_i\}_{i=1}^{\infty}$ satisfy the FSE condition, but the convergence point is IPV, then *no mechanism* can do always do better than the optimal mechanism for the IPV point (in our setting, a reserve price mechanism (Myerson 1981)).

It may not seem surprising that we cannot construct mechanisms that do well on large sets of distributions. However, the following corollary indicates that *we cannot learn* a mechanism that always does well either.

COROLLARY 2. *Let $\{\boldsymbol{\pi}_i\}_{i=1}^{\infty}$ be a sequence of distributions converging to $\boldsymbol{\pi}^*$ that satisfies Assumption 1. Denote the revenue of the optimal mechanism for the distribution $\boldsymbol{\pi}^*$ by R. For any $k>0$ and for any mechanism that is incentive compatible and individually rational and uses a finite number of independent samples from the underlying distribution, there exists a $T\in\mathbb{N}$ such that for all $\boldsymbol{\pi}_{i'}\in\{\boldsymbol{\pi}_i\}_{i=T}^{\infty}$, the expected revenue is less than $R+k$.*

It is important to be very careful in interpreting Theorem 5 and Corollary 2; they are both statements about distributions close to the convergence point. They do not provide a bound for distributions that are far from the convergence point. Therefore, even if the convergence point is an IPV distribution, it is still potentially possible to generate near optimal revenue for some distributions in the sequence, a fact we will formally show in Theorem 6. However, even with sampling, mechanisms cannot generate significantly higher revenue than the optimal IPV mechanism for distributions sufficiently close to IPV, though sampling may still substantially increase the expected revenue for some subset of the sequence of distributions.

These results indicate that the setting where the bidder may have a distribution from an infinite set is fundamentally different from the setting where the bidder's distribution is one of a finite set (as in Fu et al. (2014)). Note that the set of all mechanisms includes mechanisms that first applies some procedure to reduce the infinite set to a finite set.

In the remainder of this section, we prove Theorem 5 and Corollary 2. The strategy that we will use to prove the above results relies on bounding the maximum possible payments for any mechanism. Specifically, the revelation principle (Gibbons 1992) ensures that the revenue achievable by any mechanism can be achieved by a mechanism that not only truthfully elicits the bidder's valuation, but also truthfully elicits the distribution of the bidder. We will show that Assumption 1 implies that any mechanism with payments too large (either from or to the bidder), will create an incentive for some bidder type to lie either about his valuation or his distribution, violating the revelation principle. Once we show that payments are bounded, we can use a standard continuity result in linear programming to show that the expected revenue of the mechanism must converge to something less than or equal to the optimal revenue achievable at the convergence point.

To bound payments, we will require that for distributions "sufficiently close" to the convergence point, we can always find another distribution that is a finite step in any direction. This is what Assumption 1 provides (see Figure 3b for intuition), as the following lemma formally demonstrates.

LEMMA 2. *Let* $\{\boldsymbol{\pi}_i\}_{i=1}^{\infty}$ *be a sequence of distributions converging to* $\boldsymbol{\pi}^*$ *that satisfies Assumption 1. There exists an* $\epsilon_{\min} > 0$*, such that for all distributions* $\boldsymbol{\pi}$ *over* Ω *where* $||\boldsymbol{\pi} - \boldsymbol{\pi}^*(\cdot|\theta)|| < \epsilon_{\min}$ *for some* $\theta \in \Theta$*, and all unit vectors* $\boldsymbol{z} \in \mathbb{R}^{|\Omega|}$ *where* $\sum_{\omega} z(\omega) = 0$*, there exists a* $\boldsymbol{\pi}_j(\cdot|\theta_j) \in \{\boldsymbol{\pi}_i(\cdot|\theta)\}_{i,\theta}$ *such that* $(\boldsymbol{\pi} - \boldsymbol{\pi}_j(\cdot|\theta_j)) \bullet \boldsymbol{z} \geq \epsilon_{\min}$.

The proof is in the Appendix.

As discussed in Section 2, the payments in a Bayesian mechanism are a lottery over the external signal (see Figure 1c). A lottery over the external signal can be viewed as a linear function (or a hyper-plane) whose domain is the Ω-simplex of distributions and whose value is the expected payment for the lottery. Lemma 2 ensures that for points close enough to the convergence point, there exists a distribution in the sequence that is in the "opposite direction" of the gradient of the hyperplane that defines the lottery. I.e., for any possible lottery with a gradient of magnitude K, there exists a distribution for which the expected payment for the lottery is at least $\epsilon_{\min}K$ less than for any distributions "sufficiently close" to the convergence point. Therefore, if payments are too large (either from or to the bidder) for some distribution $\boldsymbol{\pi}'$ and θ', there is another distribution and type $\boldsymbol{\pi}''$ and θ'' that will find reporting $\boldsymbol{\pi}'$ and θ' irresistible. The following lemma formalizes this argument.

LEMMA 3. *Let* $\{\boldsymbol{\pi}_i\}_{i=1}^{\infty}$ *be a sequence of distributions converging to* $\boldsymbol{\pi}^*$ *that satisfies Assumption 1. For any mechanism* $(\boldsymbol{p}, \boldsymbol{x})$ *that is incentive compatible and individually rational and guarantees* *non-negative revenue in expectation for all distributions in* $\{\boldsymbol{\pi}_i\}_{i=1}^{\infty}$*, there exists some* $M > 0$ *such that for all* $\boldsymbol{\pi}_{i'} \in \{\boldsymbol{\pi}_i\}_{i=1}^{\infty}, \theta \in \Theta$*, and* $\omega \in \Omega$*:*

$$|x(\theta, \boldsymbol{\pi}_{i'}, \omega)| \leq M$$

The proof is in the Appendix.

With payments bounded, the final necessary result is the following stating that for any linear program where the variables for the set of optimal solutions is bounded, the corresponding sequence of linear programs is upper semi-continuous.

LEMMA 4 **(Martin (1975))**. *Let* $\boldsymbol{a}(\boldsymbol{t}), \boldsymbol{b}(\boldsymbol{t}), \boldsymbol{c}(\boldsymbol{t})$*, and* $\boldsymbol{d}(\boldsymbol{t})$ *be vectors parameterized by the parameter vector* $\boldsymbol{t} \in \mathcal{Q}$*. Assume that* $\boldsymbol{a}(\boldsymbol{t}), \boldsymbol{b}(\boldsymbol{t}), \boldsymbol{c}(\boldsymbol{t})$*, and* $\boldsymbol{d}(\boldsymbol{t})$ *converge continuously to* $\boldsymbol{a}(\boldsymbol{0}), \boldsymbol{b}(\boldsymbol{0}), \boldsymbol{c}(\boldsymbol{0})$*, and* $\boldsymbol{d}(\boldsymbol{0})$ *as* $\boldsymbol{t} \to \boldsymbol{0}$*. Similarly,* $\boldsymbol{A}(\boldsymbol{t})$*,* $\boldsymbol{B}(\boldsymbol{t})$*,* $\boldsymbol{C}(\boldsymbol{t})$*, and* $\boldsymbol{D}(\boldsymbol{t})$ *are matrices that converge continuously to* $\boldsymbol{A}(\boldsymbol{0})$*,* $\boldsymbol{B}(\boldsymbol{0})$*,* $\boldsymbol{C}(\boldsymbol{0})$*, and* $\boldsymbol{D}(\boldsymbol{0})$*.*

Define the parameterized linear program $LP(\boldsymbol{t})$ *as:*

$$\begin{aligned}
&\max_{\boldsymbol{x},\boldsymbol{q}} \quad \boldsymbol{c}'(\boldsymbol{t})\boldsymbol{x} + \boldsymbol{d}'(\boldsymbol{t})\boldsymbol{q} \\
&\textit{subject to} \\
&\boldsymbol{A}(\boldsymbol{t})\boldsymbol{x} + \boldsymbol{B}(\boldsymbol{t})\boldsymbol{q} = \boldsymbol{a}(\boldsymbol{t}) \\
&\boldsymbol{C}(\boldsymbol{t})\boldsymbol{x} + \boldsymbol{D}(\boldsymbol{t})\boldsymbol{q} \leq \boldsymbol{b}(\boldsymbol{t}) \\
&\boldsymbol{q} \geq \boldsymbol{0}
\end{aligned}$$

If the set of optimal solutions of $LP(\boldsymbol{0})$*,* $\{(\boldsymbol{x}, \boldsymbol{q}) : (\boldsymbol{x}, \boldsymbol{q}) \in \arg\max(LP(\boldsymbol{0}))\}$*, is bounded, then the objective value of* $LP(\boldsymbol{t})$ *is upper semi-continuous at* $\boldsymbol{t} = \boldsymbol{0}$*.*

All of the pieces are now in place to prove our main result, that no IC and IR mechanism over the sequence can significantly outperform the optimal mechanism for the convergence point everywhere on the sequence.

Proof of Theorem 5. Note that the maximum revenue achievable for any given $\boldsymbol{\pi}_{i'}$ can be bounded from above by the following linear program:

$$\begin{aligned}
&\max_{\boldsymbol{p},\boldsymbol{x}} \sum_{\theta} \sum_{\omega} \pi_{i'}(\theta, \omega) x(\theta, \boldsymbol{\pi}_{i'}, \omega) \\
&\textit{subject to} \\
&\sum_{\omega} \pi_{i'}(\omega|\theta) U(\theta, \boldsymbol{\pi}_{i'}, \theta, \boldsymbol{\pi}_{i'}, \omega) \geq 0 && \forall \quad \theta \in \Theta \\
&\sum_{\omega} \pi_{i'}(\omega|\theta) U(\theta, \boldsymbol{\pi}_{i'}, \theta, \boldsymbol{\pi}_{i'}, \omega) \geq \sum_{\omega} \pi_{i'}(\omega|\theta) U(\theta, \boldsymbol{\pi}_{i'}, \theta', \boldsymbol{\pi}_{i'}, \omega) && \forall \quad \theta, \theta' \in \Theta \\
&0 \leq p(\theta, \boldsymbol{\pi}_{i'}, \omega) \leq 1 && \forall \quad \theta \in \Theta, \omega \in \Omega
\end{aligned}$$

$$-M \leq x(\theta, \boldsymbol{\pi}_{i'}, \omega) \leq M \qquad \forall \quad \theta \in \Theta, \omega \in \Omega$$

where the last constraint is a consequence of Lemma 3. Therefore, by Lemma 4, the objective of this program is upper semi-continuous at $\boldsymbol{\pi}^*$, and the result follows immediately. □

Corollary 2 directly follows. The key insight is that any finite number of samples from the underlying distribution can be viewed as one signal from a more complicated distribution, and that this distribution still converges to a convergence point that will be IPV if the original convergence point is IPV.

Proof of Corollary 2 Let $\{(\theta_j, \omega_j)\}_{j=1}^N$ be a finite number of independent samples from the true distribution $\boldsymbol{\pi}_i$. Note the true distribution can be written as $\boldsymbol{\pi}_i = \boldsymbol{\pi}^* + \boldsymbol{\epsilon}_{\theta,i}$ for some $\boldsymbol{\epsilon}_{\theta,i} \in \mathbb{R}^{|\Omega|}$. Therefore, the probability of seeing samples $\{(\theta_j, \omega_j)\}_{j=1}^N$ and external signal ω is:

$$\begin{aligned}\boldsymbol{\pi}_i(\{(\theta_j, \omega_j)\}_{j=1}^N, \omega|\theta) &= \pi_i(\omega|\theta) \prod_{j=1}^N \pi_i(\omega_j|\theta_j)\pi(\theta_j) \\ &= (\pi^*(\omega|\theta) + \epsilon_{\theta,i}(\omega)) \prod_{j=1}^N (\pi^*(\omega_j|\theta_j) + \epsilon_{\theta_j,i}(\omega_j))\pi(\theta_j)\end{aligned}$$

which converges to $\boldsymbol{\pi}^*(\{(\theta_j, \omega_j)\}_{j=1}^N, \omega|\theta)$ as $\boldsymbol{\pi}_i$ converges to $\boldsymbol{\pi}^*$. Moreover, the samples $\{(\theta_j, \omega_j)\}_{j=1}^N$ are independent of the final round's bidder type, so the optimal mechanism over the distribution $\boldsymbol{\pi}^*(\{(\theta_j, \omega_j)\}_{j=1}^N, \omega|\theta)$ is revenue equivalent to the optimal mechanism over $\boldsymbol{\pi}^*$. Therefore, a finite number of samples is equivalent to a higher dimensional signal, and Theorem 5 applies directly. □

5.1. Unbounding the Approximation Ratio on a Converging Sequence of Distributions

While Corollary 2 states that we can't learn a mechanism that guarantees optimal revenue, it leaves open the possibility that we can learn nearly optimal revenue. However, as the following Lemma shows, a slight modification to Example 1 gives a sequence of distributions that all satisfy the FSE condition and whose full surplus revenue grows without bound in the number of bidder types. However, the sequence converges to an IPV distribution that has constant revenue in the number of bidder types.

Lemma 5. *In the setting of Example 1, there exists a sequence of distributions $\{\boldsymbol{\pi}_i\}_{i=1}^\infty$ that converges to an IPV distribution and satisfies Assumption 1 such that for each distribution $\boldsymbol{\pi}_i$, their exists a mechanism $(\boldsymbol{p}_i, \boldsymbol{x}_i)$ whose expected revenue is $|\Theta| + 1$.*

Proof. Let $\pi_i(\omega_L|\theta) = 1/2 + (1/i)(1/2 - (1/2)^{|\Theta| - \theta})$. Then, define the linear function

$$G_i(\boldsymbol{\pi}(\cdot|\theta)) = -\pi(\omega_L|\theta) i 2^{|\Theta|} + i 2^{|\Theta|-1} + 2^{|\Theta|-1}$$

Therefore, $G_i(\boldsymbol{\pi}_i(\cdot|\theta)) = 2^\theta$, the FSE condition, and by Theorem 3, for each $\boldsymbol{\pi}_i$, there exists a mechanism such that the expected revenue is $|\Theta|+1$. Furthermore, $\boldsymbol{\pi}_i(\cdot|\theta)$ converges to $\pi^*(\omega_L|\Theta) = 1/2$, an IPV distribution. Finally, $\pi_i(\omega_L|1) > 1/2$ while $\pi_i(\omega_L||\Theta|) < 1/2$, so the sequence satisfies Assumption 1. □

Corollary 3. *The expected revenue generated by an IR and IC mechanism over an infinite sequence of distributions guarantees at best a $(|\Theta|+1)/(2+\epsilon)$ approximation to the revenue achievable by the optimal Bayesian IC and ex-interim IR mechanism if the distribution over types is exactly known. This is still true if the mechanism designer has access to a finite number of samples from the true distribution.*

6. Computing Robust Mechanisms

Theorem 5 and Corollaries 2 and 3 show that it is impossible to provably do well for distributions near IPV, but what if we can bound the true distribution away from IPV? Can we take advantage of correlation if there is enough correlation?

6.1. Approximating the Optimal Mechanism

We now show that if the true distribution satisfies the FSE condition, then there is a single mechanism such that *for all distributions sufficiently close to the true distribution, the mechanism designer can do nearly as well as if she knew the true distribution.*

Theorem 6. *For any distribution $\boldsymbol{\pi}^*$ that satisfies the FSE condition and given any positive constant $k > 0$, there exists $\delta > 0$ and a mechanism such that for all distributions, $\boldsymbol{\pi}'$, for which for all $\theta \in \Theta$, $||\boldsymbol{\pi}^*(\cdot|\theta) - \boldsymbol{\pi}'(\cdot|\theta)|| < \delta$, the revenue generated by the mechanism is greater than or equal to $R - k$, where R is the optimal revenue for distribution $\boldsymbol{\pi}'$.*

Proof. By the assumption that there exists a mechanism that extracts full surplus for the distribution $\boldsymbol{\pi}^*$, there must be a mechanism that always allocates the item and leaves the bidder with an expected utility of 0, by Lemma 1. Let this mechanism be denoted by $(\boldsymbol{p}^*, \boldsymbol{x}^*)$. Note that this mechanism does not depend on a reported distribution, due to it being a mechanism over a single distribution. Let C be the value for the largest slope of the gradient of any lottery in the mechanism. Choose $\delta = k/(2C)$. Then the expected utility for any distribution $\boldsymbol{\pi}'(\cdot|\theta)$ with $||\boldsymbol{\pi}^*(\cdot|\theta) - \boldsymbol{\pi}'(\cdot|\theta)|| < \delta$ when optimally reporting $\theta' \in \Theta$ (note that this is, potentially, an optimal *mis*-report) is bounded by:

$$-C\delta \leq \sum_\omega \pi'(\omega|\theta) U(\theta, \theta', \omega) \leq C\delta$$

Construct a new mechanism (not necessarily truthful) where all payments $x'(\theta,\omega) = x^*(\theta,\omega) - C\delta$ and set $p'(\theta,\omega) = p^*(\theta,\omega) = 1$. Then, the utility of the bidder for optimally misreporting is:

$$0 \leq \sum_{\omega} \pi'(\omega|\theta) U(\theta,\theta',\omega) \leq 2C\delta$$

which implies that the bidder always participates. Since the item is always allocated, the loss in revenue is equivalent to the gain in utility for the bidder. Therefore, the mechanism $(\boldsymbol{p}', \boldsymbol{x}')$ always guarantees revenue within $2C\delta = k$ of the optimal mechanism for any θ, so the expected revenue of the mechanism is greater than or equal to $R - k$. $\square$

Note that the proof of Theorem 6 is constructive. Theorem 6 is intuitively very reasonable, and likely what one would expect a priori. For a class of distributions that are sufficiently close, there should be a mechanism that does about as well on all of them. However, we believe this result is a fundamental insight into the problem of correlated mechanism design; *the closeness necessary to achieve nearly full revenue is dependent on the magnitude of the largest gradient over all of the lotteries.*

Examining Figure 1c, it is intuitive that the greater the correlation, the smaller the gradient needs to be, i.e. if two points in Figure 1c are nearly on top of each other, the gradients of the lotteries necessary to distinguish between the two types is very large. Furthermore, if two valuations are far apart in magnitude (i.e. $v(\theta_1) \ll v(\theta_2)$), then the gradient of the lottery will be large.

6.2. Computing Robust Mechanisms

While the proof of Theorem 6 is constructive, and therefore, could be used to design mechanisms that achieve nearly optimal revenue, there are several problems with that approach. First, the mechanisms constructed in the proofs are not incentive compatible and individually rational, and this will lead to bidders choosing to misreport their true valuation. This both puts an additional cognitive load on the bidders to design their optimal bid, and it potentially skews any estimate of the distribution that uses the bidders report from the mechanism. Second, the mechanism is only guaranteed to be close to optimal when the set of potential distributions is quite close to the true distribution, so any mechanism designed for a case where the estimate of the true distribution is very imprecise is likely to be far from the optimal mechanism. Moreover, it is likely to perform significantly worse than even an ex-post mechanism in this setting.

Therefore, in this section, we will introduce a new mechanism design technique that can efficiently compute mechanisms that are robust incentive compatible and individually rational. This will give us strong guarantees that our mechanism will do at least as well as an ex-post mechanism while allowing for the mechanism to perform nearly optimally if the consistent set is sufficiently small.

Specifically, we will combine techniques from automated mechanism design and robust convex optimization to automate the design of robust mechanisms.

Further, while it is theoretically possible to allow bidders to report both their valuations and their beliefs, and design optimal mechanisms given this joint report, standard automated mechanism design techniques require finitely specified input, and we are explicitly interested in infinite sets of distributions. We will simplify the mechanism design process by only considering mechanisms for which the payments, $\boldsymbol{x}$, and probabilities of allocations, $\boldsymbol{p}$, depend only on the reported bidder types and the realization of the external signal. While this is not without loss of generality, it will be sufficient to achieve our goals of better than ex-post performance while allowing for the possibility of nearly optimal performance.

Definition 16 (Optimal Robust Mechanism). An *optimal robust mechanism* given an estimated distribution $\hat{\boldsymbol{\pi}}$ and a consistent set of distributions $\mathcal{P}(\hat{\boldsymbol{\pi}})$ is a mechanism that is an optimal solution to the following program:

$$\max_{\boldsymbol{x},\boldsymbol{p}} \sum_{\theta,\omega} \hat{\pi}(\theta,\omega)x(\theta,\omega) \tag{5}$$

subject to

$$\sum_{\omega\in\Omega} \pi'(\omega|\theta)U(\theta,\theta,\omega) \geq 0 \qquad \forall \quad \theta\in\Theta,\, \boldsymbol{\pi}'(\cdot|\theta)\in\mathcal{P}(\hat{\boldsymbol{\pi}}(\cdot|\theta)) \tag{6}$$

$$\sum_{\omega\in\Omega} \pi'(\omega|\theta)U(\theta,\theta,\omega) \geq \sum_{\omega\in\Omega} \pi'(\omega|\theta)U(\theta,\theta',\omega) \qquad \forall \quad \theta,\theta'\in\Theta,\, \boldsymbol{\pi}'(\cdot|\theta)\in\mathcal{P}(\hat{\boldsymbol{\pi}}(\cdot|\theta)) \tag{7}$$

$$0 \leq p(\theta,\omega) \leq 1 \qquad \forall \quad \theta\in\Theta,\, \omega\in\Omega \tag{8}$$

Note that the linear program in Definition (16) still contains an infinite number of constraints over a, potentially, non-convex set, and therefore is in general computationally intractable. However, the following assumption allows computational tractability.

Assumption 2. *The set $\mathcal{P}(\hat{\boldsymbol{\pi}})$ is such that for all $\theta\in\Theta$, $\mathcal{P}(\hat{\boldsymbol{\pi}}(\cdot|\theta))$ is a convex n-polyhedron where n is polynomial in the number of bidder types.*

Assumption 2 includes very reasonable cases such as the case where for all $\theta\in\Theta$ and $\omega\in\Omega$, $\pi'(\omega|\theta)\in[\underline{\pi}(\omega|\theta),\overline{\pi}(\omega|\theta)]$. Further, any set that does not satisfy Assumption 2 can be contained in a set that does. Therefore, we can always make the assumption hold by using a larger consistent set.

Theorem 7. *For a given $(\boldsymbol{p},\boldsymbol{x})$ and $\mathcal{P}(\hat{\boldsymbol{\pi}})$ that satisfy Assumption 2, there exists a polynomial time algorithm that determines whether there exists a $\boldsymbol{\pi}'(\cdot|\theta)\in\mathcal{P}(\hat{\boldsymbol{\pi}}(\cdot|\theta))$ such that robust individual rationality or robust incentive compatibility is violated.*

Proof. For each $\theta \in \Theta$, solve the following linear program

$$\begin{aligned} &\min_{\boldsymbol{\pi}'(\cdot|\theta)} \sum_{\omega} \pi'(\omega|\theta)(v(\theta)p(\omega,\theta) - x(\omega,\theta)) \\ &\textit{subject to} \\ &\boldsymbol{\pi'}(\cdot|\theta) \in \mathcal{P}(\hat{\boldsymbol{\pi}}(\cdot|\theta)) \end{aligned} \tag{9}$$

Note that in the program (9), $(\boldsymbol{p}, \boldsymbol{x})$ are no longer variables but coefficients. If (9) has an objective value of less than 0, then the robust IR constraint with distribution $\boldsymbol{\pi}'$ is violated. If the objective value is at least 0, there is no robust IR constraint violated for θ.

There are $|\Theta|$ linear programs that must be solved, each with a polynomial number of variables and constraints, due to Assumption 2. Therefore, violated robust IR constraints can be generated in polynomial time.

Similarly for robust incentive compatibility, the following program, for all $\theta, \theta' \in \Theta$, finds violated constraints:

$$\begin{aligned} &\min_{\boldsymbol{\pi}'(\cdot|\theta)} \sum_{\omega} \pi'(\omega|\theta)\left(v(\theta)p(\omega,\theta) - x(\omega,\theta) - (v(\theta)p(\omega,\theta') - x(\omega,\theta'))\right) \\ &\textit{subject to} \\ &\boldsymbol{\pi'}(\cdot|\theta) \in \mathcal{P}(\hat{\boldsymbol{\pi}}(\cdot|\theta)) \end{aligned}$$

□

Corollary 4. *If $\mathcal{P}(\hat{\boldsymbol{\pi}})$ satisfies Assumption 2, the optimal robust mechanism can be computed in time polynomial in the number of types of the bidder and external signal.*

Proof. By Theorem 7, we can determine whether or not a robust IR or robust IC constraint is violated in polynomial time, and add the constraint to the linear program. There are $2|\Theta||\Omega|$ variables in the linear program in Definition 16, and there are $2|\Theta||\Omega|$ non-IC and IR constraints.

Therefore, by the ellipsoid method, the optimal robust mechanism can be computed in polynomial time (Kozlov et al. 1980). □

Note that Corollary 4 states that the optimal robust mechanism is polynomial in the *number of bidder types*, not the number of bidders. The current formulation is exponential in the number of bidders, since the distribution is exponential in the number of bidders. For a distribution that has support that is non-exponential in the number of bidders, the optimal robust mechanism can be computed in polynomial time in the number of bidders. However, as stated in the introduction, much of the advantage of prior-dependent mechanisms will be in thin auctions, so we do not view this as a significant weakness of this approach.

Since for ex-post mechanisms, incentive compatibility and individual rationality are independent of the distribution, it would be expected that when we have no useful information about the distribution, the optimal robust mechanism should be equivalent to the optimal ex-post mechanism. The following corollary shows that this is indeed the case.

COROLLARY 5. *If for all $\theta \in \Theta$, and $\omega \in \Omega$, $\mathcal{P}(\hat{\boldsymbol{\pi}})$ is such that the distribution $\pi'(\omega'|\theta) = 1$ if $\omega' = \omega$ and 0 otherwise is in $\mathcal{P}(\hat{\boldsymbol{\pi}})$, then an optimal robust mechanism is an optimal ex-post mechanism for the distribution $\hat{\boldsymbol{\pi}}$.*

Proof. If for all $\theta \in \Theta$ and $\omega \in \Omega$, the distribution such that $\pi'(\omega'|\theta) = 1$ if $\omega' = \omega$ and 0 otherwise is in $\mathcal{P}(\hat{\boldsymbol{\pi}})$, the robust IR constraints contain the following set of constraints

$$v(\theta)p(\theta,\omega) - x(\theta,\omega) \geq 0 \qquad \forall \quad \omega \in \Omega, \theta \in \Theta$$

which implies ex-post IR. Conversely, ex-post IR implies robust IR.

By an identical argument, the robust IC constraints imply the ex-post IC constraints, and vice-versa. □

6.3. ϵ-Robust Mechanisms

While so far we have been assuming that there is a well defined set, $\mathcal{P}(\hat{\boldsymbol{\pi}})$, such that the mechanism designer can guarantee that the true distribution, $\boldsymbol{\pi}$, is in the set, this is unlikely to be a realistic assumption in practice. It is far more reasonable that the mechanism designer would have a set such that the true distribution is in the set with high probability. If this is the case, we can still design mechanisms that are likely to outperform weakly prior dependent mechanisms, such as ex-post mechanisms, by relaxing the requirement that the mechanism be always incentive compatible and always individually rational. We will define the set of ϵ-consistent distributions as follows.

DEFINITION 17 (SET OF ϵ-CONSISTENT DISTRIBUTIONS). A subset $\mathcal{P}_\epsilon(\hat{\boldsymbol{\pi}}) = \mathcal{P}_\epsilon(\{\hat{\boldsymbol{\pi}}_\theta, \hat{\boldsymbol{\pi}}(\cdot|\cdot)\}) \subseteq P(\Theta) \times \prod_{\theta\in\Theta} P(\Omega)$ is an *ϵ-consistent set of distributions* for the estimated distribution $\hat{\boldsymbol{\pi}} = \{\hat{\boldsymbol{\pi}}_\theta, \hat{\boldsymbol{\pi}}(\cdot|\cdot)\}$ if the true distribution, $\boldsymbol{\pi} = \{\boldsymbol{\pi}_\theta, \boldsymbol{\pi}(\cdot|\cdot)\}$, is in $\mathcal{P}_\epsilon(\hat{\boldsymbol{\pi}})$ with probability $1-\epsilon$.

Now we can define the notion of *ϵ-robust individual rationality* and *incentive compatibility.* These definitions are analogous to Definitions 12 and 13.

DEFINITION 18 (ϵ-ROBUST INDIVIDUAL RATIONALITY). A mechanism is *ϵ-robust individually rational* for estimated bidder distribution $\hat{\boldsymbol{\pi}}$ and ϵ-consistent set of distributions $\mathcal{P}_\epsilon(\hat{\boldsymbol{\pi}})$ if for all $\theta \in \Theta$ and $\boldsymbol{\pi} \in \mathcal{P}_\epsilon(\hat{\boldsymbol{\pi}})$,

$$\sum_{\omega\in\Omega} \pi(\omega|\theta) U(\theta, \boldsymbol{\pi}, \theta, \boldsymbol{\pi}, \omega) \geq 0$$

Definition 19 (ϵ-Robust Incentive Compatibility). A mechanism is *ϵ-robust incentive compatible* for estimated bidder distribution $\hat{\boldsymbol{\pi}}$ and ϵ-consistent set of distributions $\mathcal{P}_\epsilon(\hat{\boldsymbol{\pi}})$ if for all $\theta, \theta' \in \Theta$ and $\boldsymbol{\pi}, \boldsymbol{\pi}' \in \mathcal{P}_\epsilon(\hat{\boldsymbol{\pi}})$,

$$\sum_{\omega \in \Omega} \pi(\omega|\theta) U(\theta, \boldsymbol{\pi}, \theta, \boldsymbol{\pi}, \omega) \geq \sum_{\omega \in \Omega} \pi(\omega|\theta) U(\theta, \boldsymbol{\pi}, \theta', \boldsymbol{\pi}', \omega)$$

Similarly to Definition 16, we can define the optimal ϵ-robust mechanism. Again, for tractability of mechanism design, and to obtain more practical mechanisms, we will restrict attention to mechanisms that only depend on the reported bidder type and the external signal.

Definition 20 (Optimal ϵ-Robust Mechanism). An *optimal ϵ-robust mechanism* given an estimated distribution $\hat{\boldsymbol{\pi}}$ and an ϵ-consistent set of distributions $\mathcal{P}_\epsilon(\hat{\boldsymbol{\pi}})$ is a mechanism that is an optimal solution to the following program:

$$\begin{aligned}
&\max_{x(\theta,\omega), p(\theta,\omega)} \sum_{\theta,\omega} \hat{\pi}(\theta,\omega) x(\theta,\omega) \\
&\text{subject to} \\
&\sum_{\omega \in \Omega} \pi'(\omega|\theta) U(\theta,\theta,\omega) \geq 0 && \forall \quad \theta \in \Theta,\, \boldsymbol{\pi}'(\cdot|\theta) \in \mathcal{P}_\epsilon(\hat{\boldsymbol{\pi}}(\cdot|\theta)) \\
&\sum_{\omega \in \Omega} \pi'(\omega|\theta) U(\theta,\theta,\omega) \geq \sum_{\omega \in \Omega} \pi'(\omega|\theta) U(\theta,\theta',\omega) && \forall \quad \theta,\theta' \in \Theta,\, \boldsymbol{\pi}'(\cdot|\theta) \in \mathcal{P}_\epsilon(\hat{\boldsymbol{\pi}}(\cdot|\theta)) \\
&0 \leq p(\theta,\omega) \leq 1 && \forall \quad \theta \in \Theta,\, \omega \in \Omega
\end{aligned}$$

Corollary 6. *If $\mathcal{P}_\epsilon(\hat{\boldsymbol{\pi}})$ satisfies Assumption 2, the optimal ϵ-robust mechanism can be computed in time polynomial in the number of bidder types and external signals.*

So far, we have viewed ϵ as a consequence of the estimation procedure. However, it is equally valid to view ϵ as a parameter indicating the level of robustness desired by the mechanism. For example, a non-parametric estimation procedure with an $\epsilon = 0$ will return a consistent set equal to all possible distributions. The optimal ϵ-robust mechanism will then correspond to the optimal ex-post mechanism (Corollary 5) with the estimated distribution $\hat{\boldsymbol{\pi}}$. Similarly, if $\epsilon = 1$, then the optimal ϵ-robust mechanism will correspond to the optimal Bayesian mechanism, since the consistent set will consist of the singleton $\hat{\boldsymbol{\pi}}$. Therefore, the parameter ϵ can be viewed as a regularization parameter for learning optimal mechanisms under correlated valuation settings. Moreover, there is nothing in the definition of the optimal ϵ-robust mechanism that precludes a separate ϵ for IR versus IC or even different bidder types within the IC or IR constraint set. In principle, different levels of regularization could be applied for different bidder types. This feature will be used in computing the sample complexity in Section 6.4.

6.4. Sample Complexity of ϵ-Robust Mechanisms

The ϵ-robust mechanism design procedure proposed in Section 6.3 is well defined for any ϵ-consistent set, and the performance of an ϵ-robust mechanism will exceed that of ex-post mechanisms, with high probability, for very large consistent sets. Moreover, it will be incentive compatible with high probability as well. However, we have not demonstrated that the expected revenue from the mechanism will converge to the optimal revenue for a sufficiently small ϵ-consistent set, nor have we demonstrated a guarantee that this approach uses samples efficiently.

In this section, we show, using an ϵ-consistent set defined by a *non-parametric* estimation of the true distribution, that the expected revenue from an optimal ϵ-robust mechanism converges to the revenue achievable under full surplus extraction, assuming that the true underlying distribution satisfies the FSE condition. Moreover, we show that we require only a polynomial number of samples from the underlying distribution to achieve an additive k-approximation, where the sample complexity is over the number of bidder types, the number of external signals, the largest valuation $v(|\Theta|)$, and the separation in conditional beliefs of the bidder.

We will restrict this analysis to the case where the true underlying distribution satisfies the FSE condition. This is due to the absence of a general characterization of optimal mechanisms in correlated valuation settings when the FSE condition does not hold. However, we hope that this work, particularly results from Section 3, provides insight into the optimal mechanism for correlated distributions that do not satisfy the FSE condition, and if so, we expect some elements of the sample complexity results presented here to carry over to bounding the general approximation error.

As we know from Corollary 3, for an arbitrary distribution, there is no mechanism based on sampling that will guarantee a constant approximation. Therefore, we require that the true distribution not only satisfies the FSE condition, but that it satisfies the FSE condition by a sufficient margin. The following definition provides a condition that formalizes this.

Definition 21 (γ-Separated Consistent Set). For a consistent set, $\mathcal{P}(\hat{\boldsymbol{\pi}})$, and for all subsets $\hat{\Theta} \subset \Theta$, let

$$Conv\{\hat{\Theta}\} = \left\{ \boldsymbol{\pi}' \in P(\Omega) \,\middle|\, \exists \{\boldsymbol{\pi}_i\}_{i \in \Omega} \subset \bigcup_{\theta \in \hat{\Theta}} \mathcal{P}(\hat{\boldsymbol{\pi}}(\cdot|\theta)) \wedge \alpha_i \geq 0 \wedge \sum_{i \in \Omega} \alpha_i = 1 \; s.t. \sum_{i \in \Omega} \alpha_i \boldsymbol{\pi}_i = \boldsymbol{\pi}' \right\}. \quad (10)$$

I.e., $Conv\{\hat{\Theta}\}$ is the convex hull over all conditional distributions, for $\theta \in \hat{\Theta}$, in the consistent set.

A consistent set is said to be γ*-separated* if for all $\hat{\Theta}$ such that $|\hat{\Theta}| = \max\{|\Theta| - 1, |\Omega| - 1\}$ and all $\theta \in \Theta \setminus \hat{\Theta}$

$$\gamma \leq \min_{\boldsymbol{\pi} \in Conv\{\hat{\Theta}\},\, \boldsymbol{\pi}' \in Conv\{\{\theta\}\}} ||\boldsymbol{\pi} - \boldsymbol{\pi}'||. \quad (11)$$

Moreover, a distribution $\boldsymbol{\pi}$ can be viewed as a consistent set that is just a singleton, so a γ-separated distribution is similarly defined.

If a consistent set is γ-separated, then every distribution in the consistent set is also γ-separated. Therefore, if the true distribution is in the consistent set, then it is also γ-separated. Using the notion of a γ-separated consistent set, we can bound the payments of an optimal robust mechanism.

LEMMA 6. *If the consistent set, $\mathcal{P}(\hat{\boldsymbol{\pi}})$, is γ-separated for $\gamma > 0$, then there exists an optimal robust mechanism $(\boldsymbol{p}, \boldsymbol{x})$ such that for all $\theta \in \Theta$ and $\omega \in \Omega$,*

$$-\frac{v(|\Theta|)}{\gamma} \leq x(\theta,\omega) \leq \frac{v(|\Theta|)}{\gamma} + v(\Theta) \leq \frac{2v(|\Theta|)}{\gamma}. \tag{12}$$

The proof is in the Appendix.

Lemma 6 allows an upper bound on the loss of the ϵ-robust mechanism when the mechanism fails to include the true distribution in the consistent set. The worst possible outcome for the mechanism designer, in the case where the true distribution is not in the consistent set, would be for a bidder type that would have faced a payment of $\frac{2v(|\Theta|)}{\gamma}$ to, instead, misreport and face a payment of $-\frac{v(|\Theta|)}{\gamma}$. Therefore, the loss for a bidder type that misreports is less than $\frac{2v(|\Theta|)}{\gamma} - \left(-\frac{v(|\Theta|)}{\gamma}\right) = \frac{3v(|\Theta|)}{\gamma}$. Going forward, we will add this additional non-binding (by Lemma 6) constraint to the programs that define the optimal robust (Definition 16) and ϵ-robust (Definition 20) mechanisms for γ-consistent sets:

$$-\frac{v(|\Theta|)}{\gamma} \leq x(\theta,\omega) \leq \frac{2v(|\Theta|)}{\gamma} \qquad \forall \theta \in \Theta, \omega \in \Omega \tag{13}$$

Note that this constraint does not apply to the case where the consistent set is not γ-separated, i.e $\gamma = 0$. This allows a guarantee that, for a sufficiently small consistent set, the robust mechanism does indeed converge to the optimal mechanism, as the following lemma demonstrates.

LEMMA 7. *Let $\mathcal{P}(\hat{\boldsymbol{\pi}})$ be a γ-separated consistent set such that for all $\theta \in \Theta$ there exists a $\delta > 0$ such that $\delta \geq \max_{\boldsymbol{\pi},\boldsymbol{\pi}' \in \mathcal{P}(\hat{\boldsymbol{\pi}}(\cdot|\theta))} ||\boldsymbol{\pi} - \boldsymbol{\pi}'||$, $\delta \geq \max_{\boldsymbol{\pi},\boldsymbol{\pi}' \in \mathcal{P}(\hat{\boldsymbol{\pi}})} ||\boldsymbol{\pi} - \boldsymbol{\pi}'||$, and $\delta \leq \frac{k\gamma}{6v(|\Theta|)|\Omega|}$. Let $\boldsymbol{\pi}^* \in \mathcal{P}(\hat{\boldsymbol{\pi}})$ be a distribution that satisfies the FSE condition with optimal revenue R. Then an optimal robust mechanism achieves at least $R - k$ in expected revenue for any $\boldsymbol{\pi} \in \mathcal{P}(\hat{\boldsymbol{\pi}})$, where R is the optimal revenue for the distribution $\boldsymbol{\pi}^*$.*

The proof is in the Appendix.

Now that we have bounded the loss in revenue due to using a robust mechanism, we need two additional pieces to prove our main result. First, we need an upper bound on the number of samples from the underlying distribution necessary to make the ϵ-consistent set sufficiently small with a sufficiently small ϵ. Second, we must bound the loss due to the, less than ϵ, probability that the true distribution is not in the consistent set. We will bound the number of samples next, and to do so, we will rely on two concentration inequalities.

Lemma 8 **(Devroye (1983))**. *Let $(\theta_1, ..., \theta_k)$ be a multinomial $(n, p_1, ..., p_k)$ random vector. For all $\delta \in (0,1)$ and all k satisfying $\frac{k}{n} \leq \frac{\delta^2}{20}$, we have*

$$P\left(\sum_{i=1}^{k} |\theta_i - E(\theta_i)| > n\delta\right) \leq 3e^{\frac{-n\delta^2}{25}} \tag{14}$$

Lemma 9 **(Mitzenmacher and Upfal (2005))**. *Let θ be a binomial (n,p) random variable. For all $\delta \in (0,1)$, we have*

$$P(\theta < (1-\delta)np) \leq e^{-\frac{\delta^2 np}{2}} \tag{15}$$

Note that we will use $\hat{\boldsymbol{\pi}}_X$ to denote the empirical distribution function from seeing the set of samples X. I.e., $\hat{\boldsymbol{\pi}}_X(\theta,\omega) = \frac{1}{|X|}\sum_{x\in X} \mathbb{1}_{\{x\equiv(\theta,\omega)\}}$. We will refer to the subset of samples in which the bidder type is $\theta \in \Theta$ as X_θ.

Lemma 10. *Let $\boldsymbol{\pi}$ be the true distribution over $\Theta \times \Omega$, and let X be a set of independent samples from $\boldsymbol{\pi}$. For $\theta \in \Theta$, if $|X_\theta| \geq 25|\Omega| \ln\left(\frac{1}{\epsilon_1}\right)\left(\frac{1}{\delta_1}\right)^2$, then $||\hat{\boldsymbol{\pi}}_X(\cdot|\theta) - \boldsymbol{\pi}(\cdot|\theta)|| \leq \delta_1$ with probability $1-\epsilon_1$.*

Proof. By Devroye's Lemma (Lemma 8), with $\epsilon_1 = 3e^{\frac{-|X_\theta|\delta_1^2}{25}}$, the number of samples necessary is:

$$|X_\theta| \geq 25 \ln\left(\frac{1}{\epsilon_1}\right)\left(\frac{1}{\delta_1}\right)^2.$$

However, $\frac{|\Omega|}{|X_\theta|} \leq \frac{\delta_1^2}{20}$ implying:

$$|X_\theta| \geq 25|\Omega| \ln\left(\frac{1}{\epsilon_1}\right)\left(\frac{1}{\delta_1}\right)^2 \geq \max\left\{25 \ln\left(\frac{1}{\epsilon_1}\right)\left(\frac{1}{\delta_1}\right)^2, \frac{20|\Omega|}{\delta_1^2}\right\} \tag{16}$$

is sufficient. □

Lemma 10 bounds the number of samples necessary to estimate the conditional distribution for a particular $\theta \in \Theta$. However, those samples must be samples in X_θ. Lemma 11 bounds the number of samples, X, necessary to ensure that the mechanism designer sees a sufficient number of samples of type $\theta \in \Theta$. However, we do not put a lower bound on the marginal probability of a bidder type θ, so there will be, in general, no finite number of samples sufficient to ensure that the number of samples with bidder type θ is sufficient to estimate the conditional distribution. Therefore, we bound the probability mass of the bidder types for which we will not see a sufficient number of samples.

Lemma 11. *Let $\boldsymbol{\pi}$ be the true distribution over $\Theta \times \Omega$, and let X be a set of independent samples from $\boldsymbol{\pi}$. Let $\Theta' = \{\theta | \theta \in \Theta, |X_\theta| \geq M\}$. Then, if $|X| \geq 8M|\Theta| \ln\left(\frac{|\Theta|}{\epsilon_2}\right)\left(\frac{1}{\delta_2}\right)$, $\sum_{\theta\in\Theta'} \pi(\theta) > 1 - \delta_2$ with probability $1-\epsilon_2$.*

Proof. For any sample, we can model the probability of that sample being of bidder type $\theta \in \Theta$ as a binomial distribution with success probability of $\pi(\theta)$. In order to ensure that $\sum_{\theta \in \Theta'} \pi(\theta) > 1 - \delta_2$ with probability $1 - \epsilon_2$, we need to cover any bidder with type $\frac{\delta_2}{|\Theta|}$ marginal probability with probability at least $1 - \frac{\epsilon_2}{|\Theta|}$. If we do this, then there is at most a set of bidders with collective marginal probability of less than δ_2 that we do not cover, with probability of at least $1 - \epsilon_2$. The number of samples we need to ensure that we cover a bidder with marginal probability of $\frac{\delta_2}{|\Theta|}$ with probability $1 - \frac{\epsilon_2}{|\Theta|}$ is given by Lemma 9. Set $\delta = \frac{1}{2}$ in Lemma 9, then $\frac{\epsilon_2}{|\Theta|} = e^{-\frac{|X|\delta_2}{8|\Theta|}}$:

$$|X| \geq 8|\Theta| \ln\left(\frac{|\Theta|}{\epsilon_2}\right)\left(\frac{1}{\delta_2}\right) \tag{17}$$

and $M \leq \frac{|X|\delta_2}{2|\Theta|}$ which implies:

$$|X| \geq 8M|\Theta| \ln\left(\frac{|\Theta|}{\epsilon_2}\right)\left(\frac{1}{\delta_2}\right) \geq \max\left\{8|\Theta| \ln\left(\frac{|\Theta|}{\epsilon_2}\right)\left(\frac{1}{\delta_2}\right), \frac{2M|\Theta|}{\delta_2}\right\} \tag{18}$$

□

Finally, we must ensure that we estimate the objective sufficiently well.

LEMMA 12. *Let $\boldsymbol{\pi}$ be the true distribution over $\Theta \times \Omega$, and let X be a set of independent samples from $\boldsymbol{\pi}$. Then, if $|X| \geq 25|\Omega||\Theta| \ln\left(\frac{1}{\epsilon_3}\right)\left(\frac{1}{\delta_3}\right)^2$, $||\hat{\boldsymbol{\pi}}_X - \boldsymbol{\pi}|| \leq \delta_3$ with probability $1 - \epsilon_3$.*

Proof. Proof is identical to Lemma 10. □

Combining the above results, we can achieve our main result, a polynomial bound on the number of samples necessary to achieve a k-additive approximation to the optimal expected revenue when the distribution is exactly known. Note that we bound the expected revenue of the ϵ-robust mechanism in Theorem 8, and this expectation is over both the performance of the mechanism given the distribution over bidder types and external signals *and the distribution over the samples that we may observe*. The loss in expected revenue due to each of these sources of uncertainty is bounded separately in the proof of Theorem 8.

THEOREM 8. *Let the true distribution, $\boldsymbol{\pi}^*$, be a γ-separated distribution that satisfies the FSE condition. Let X be independent samples from $\boldsymbol{\pi}^*$. If $|X| \geq 50|\Omega|^3|\Theta| \ln^2(\frac{24|\Theta|v(|\Theta|)}{k\gamma})(\frac{48v(|\Theta|)}{k\gamma})^3$, then an optimal ϵ-robust mechanism where the estimated distribution is $\hat{\boldsymbol{\pi}}_X$ and the consistent set $\mathcal{P}_\epsilon(\hat{\boldsymbol{\pi}}_X) \subseteq \left\{\boldsymbol{\pi} \,\middle|\, ||\boldsymbol{\pi} - \hat{\boldsymbol{\pi}}_X|| \leq \frac{k\gamma}{48v(|\Theta|)|\Omega|}\right\}$ and $\mathcal{P}_\epsilon(\hat{\boldsymbol{\pi}}_X(\cdot|\theta)) \subseteq \left\{\boldsymbol{\pi}(\cdot|\theta) \,\middle|\, ||\boldsymbol{\pi}(\cdot|\theta) - \hat{\boldsymbol{\pi}}_X(\cdot|\theta)|| \leq \frac{k\gamma}{48v(|\Theta|)|\Omega|}\right\}$, has an expected revenue of at least $R - k$, where R is the optimal revenue achievable with $\boldsymbol{\pi}^*$.*

Proof. There are two main ways in which the optimal mechanism can fail to achieve full revenue. First, if the true distribution is in the consistent set, the mechanism loses some revenue due to the robust constraints and mis-estimation of the objective function (this is bounded by Lemma 7).

Second, if the true distribution is not in the consistent set, then the mechanism can lose revenue due to bidders misreporting, or reporting accurately but the objective is very far from correct.

First, we will consider the loss if the true distribution is in $\mathcal{P}_\epsilon(\hat{\boldsymbol{\pi}}_X)$. By Lemma 7, if the set of possible distribution is small enough, i.e. if $\delta < \frac{k\gamma}{24v(|\Theta|)|\Omega|}$, then the expected revenue of the mechanism is at least $R - \frac{k}{4}$. Note that our definition of $\mathcal{P}_\epsilon(\hat{\boldsymbol{\pi}}_X)$ is one sided, so we must have $\mathcal{P}_\epsilon(\hat{\boldsymbol{\pi}}_{X,\theta}) \subseteq \left\{\boldsymbol{\pi}_\theta \,\middle|\, ||\boldsymbol{\pi}_\theta - \hat{\boldsymbol{\pi}}_{X,\theta}|| \leq \frac{\delta}{2} = \frac{k\gamma}{48v(|\Theta|)|\Omega|}\right\}$, and similarly for $\mathcal{P}_\epsilon(\hat{\boldsymbol{\pi}}_X(\cdot|\theta))$.

If the true distribution is not in the consistent set, then the worst that can happen is that a bidder type that would have made a maximal payment, instead makes a minimal payment, a loss of revenue bounded by $\frac{3v(|\Theta|)}{\gamma}$ by Lemma 6. The true distribution can be such that the full distribution is not in $\mathcal{P}_\epsilon(\hat{\boldsymbol{\pi}}_X)$ or the conditional distribution for the bidder's type θ may not be in $\mathcal{P}_\epsilon(\hat{\boldsymbol{\pi}}_X(\cdot|\theta))$. However, the probability for each of these events happening, independently, is less than ϵ, so the probability of either happening for a specific bidder type $\theta \in \Theta$ is less than 2ϵ, by a union bound. Therefore, if $\epsilon \leq \frac{k\gamma}{24v(|\Theta|)}$, the revenue loss due to the distribution not being in the consistent set is at most $\frac{k}{4}$.

There is one other additional source of potential loss. Specifically, for a bidder type $\theta' \in \Theta$, if the marginal probability, $\pi_\theta(\theta')$, of that type is too low, then it will be infeasible to estimate the conditional distribution for that type. Define the consistent set for the conditional distribution for θ to be $\mathcal{P}_{\epsilon'}(\hat{\boldsymbol{\pi}}_X(\cdot|\theta')) \subseteq \left\{\boldsymbol{\pi}(\cdot|\theta') \,\middle|\, ||\boldsymbol{\pi}(\cdot|\theta') - \hat{\boldsymbol{\pi}}_X(\cdot|\theta')|| \leq \frac{k\gamma}{48v(|\Theta|)|\Omega|}\right\}$. However, ϵ' may be large, as high as 1. This is a slight abuse of notation, as now $\epsilon' \neq \epsilon$ for a subset of bidder types. As discussed, the definition of the optimal ϵ-robust mechanism (Definition 20) does not require ϵ to be the same for all $\theta \in \Theta$, so the mechanism is still well defined. In this case, we assume that $\epsilon' = 1$, and we lose the maximum possible revenue, $\frac{3v(|\Theta|)}{\gamma}$, for every bidder type θ' for which we do not accurately estimate the conditional distribution. Therefore, if we ensure that the probability mass of bidder types such that this happens is less than $\frac{k\gamma}{12v(|\Theta|)}$ with high probability, we can bound the loss. Then if the probability mass is less than $\frac{k\gamma}{12v(|\Theta|)}$, we lose at most $\frac{k}{4}$ in revenue. Further, we bound the probability that the probability mass is larger than $\frac{k\gamma}{12v(|\Theta|)}$ by, again, $\frac{k\gamma}{12v(|\Theta|)}$. If the probability mass is more than $\frac{k\gamma}{12v(|\Theta|)}$, we assume that we lose the maximum possible amount, $\frac{3v(|\Theta|)}{\gamma}$, implying that the loss in revenue in expectation is less than $\frac{k}{4}$ for the case where there are too many bidder types for which the estimate of the conditional distribution is insufficient.

Therefore, if the number of samples is sufficient to ensure the above conditions, the expected revenue from the mechanism is greater than or equal to $R - k$.

To calculate a sufficient number of samples, we use Lemmas 10, 11, and 12. Using the notation of the Lemmas, we must have $\delta_1 = \delta_3 = \delta/2 \leq \frac{k\gamma}{48v(|\Theta|)|\Omega|}$, $\epsilon_1 = \epsilon_3 = \epsilon \leq \frac{k\gamma}{24v(|\Theta|)}$, $\epsilon_2 \leq \frac{k\gamma}{12v(|\Theta|)}$, and $\delta_2 \leq \frac{k\gamma}{12v(|\Theta|)}$.

Therefore, $|X| \geq 25|\Omega||\Theta|\ln(\frac{24v(|\Theta|)}{k\gamma})(\frac{48v(|\Theta|)|\Omega|}{k\gamma})^2$ by Lemma 12. Also, by Lemmas 10 and 11,

$$\begin{aligned}|X| &\geq 8M|\Theta|\ln\left(\frac{12|\Theta|v(|\Theta|)}{k\gamma}\right)\frac{12v(|\Theta|)}{k\gamma}\\ &= 50|\Omega|\ln\left(\frac{24v(|\Theta|)}{k\gamma}\right)\left(\frac{48v(|\Theta|)|\Omega|}{k\gamma}\right)^2|\Theta|\ln\left(\frac{12|\Theta|v(|\Theta|)}{k\gamma}\right)\frac{48v(|\Theta|)}{k\gamma}\\ &= 50|\Omega|^3|\Theta|\ln\left(\frac{24v(|\Theta|)}{k\gamma}\right)\ln\left(\frac{12|\Theta|v(|\Theta|)}{k\gamma}\right)\left(\frac{48v(|\Theta|)}{k\gamma}\right)^3.\end{aligned}$$

Therefore, if $|X| \geq 50|\Omega|^3|\Theta|\ln^2(\frac{24|\Theta|v(|\Theta|)}{k\gamma})(\frac{48v(|\Theta|)}{k\gamma})^3$, both conditions are satisfied. □

Note that, as is common in sample complexity results, Theorem 8 gives a very loose upper bound on the number of samples necessary. Many of the simplifying steps in the proof increased the number of samples by significant factors, and a more complicated analysis would likely significantly reduce this bound. Moreover, as we show in Section 6.5, with a naïve estimation procedure, we can generate nearly optimal revenue with many fewer samples. However, Theorem 8 formally demonstrates that this estimation problem, even in the worst case, is not intractable.

Corollary 7. *The sample complexity for constructing an ϵ-robust mechanism with an expected revenue that is an additive k-approximation to the expected revenue of the optimal mechanism over a true distribution that is γ-separated and satisfies the FSE condition is poly*$(\frac{1}{k}, \frac{1}{\gamma}, |\Theta|, |\Omega|, v(|\Theta|))$.

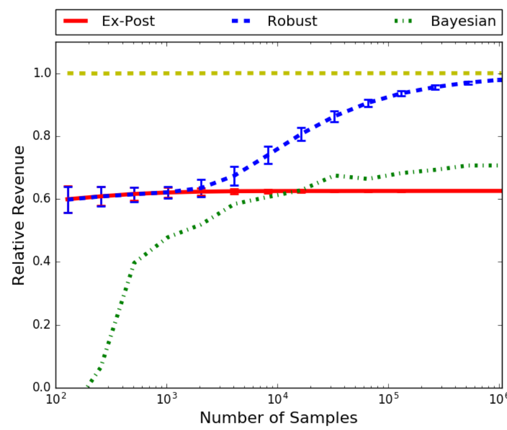
6.5. Experimental Results

Throughout the experiments, we have a single bidder with type $\theta \in \{1, 2, ..., 10\}$ and valuation $v(\theta) = \theta$. The external signal is $\omega \in \{1, 2, ..., 10\}$. We model the true distribution as a categorical distribution with 10×10 elements, with each element corresponding to a tuple (θ, ω).

There are not, to our knowledge, standard distributions to test correlated mechanism design procedures available, so we use a discretized bi-variate normal distribution. Specifically, we discretize the area under the bi-variate standard normal distribution between $[-1.96, 1.96]$ in both dimensions as a 10×10 grid and normalize. We chose the bi-variate normal distribution for its broad relevance to many empirically observed distributions and the ability to easily vary the correlation. Note that the bi-variate normal distribution always satisfies the Cremer-McLean condition if the correlation is positive.

To estimate the distribution, we sample from the true distribution and use Bayesian updating with a maximally uninformative Dirichlet prior ($\alpha = [1, ..., 1]$) to arrive at a Dirichlet posterior over the distribution of bidder types and external signals. We then calculate empirical confidence intervals by sampling from the Dirichlet posterior and observing the $\epsilon/(2*10*10)$ and $(1 - \epsilon/(2*10*10))$ quantiles for each element of the conditional distributions $\boldsymbol{\pi}(\omega|\theta)$ and use the quantiles as the ϵ-consistent set. Note that we do not simply use the $\epsilon/2$ and $(1 - \epsilon/2)$ quantiles due to jointly

Figure 4 The performance of the ex-post, ϵ-robust, and Bayesian mechanisms using the estimated distribution. All revenue is scaled by the full social surplus, denoted as 1. Note that the Number of Samples is in log scale. The parameters used were as follows: Correlation = .5, ϵ = .05. Each experiment was repeated 200 times, and the 95% confidence interval is included for the ϵ-robust and ex-post mechanisms. The Bayesian mechanism confidence interval is off the plot.



estimating confidence intervals for 100 variables; the expression for the quantiles above is based on applying a union bound.

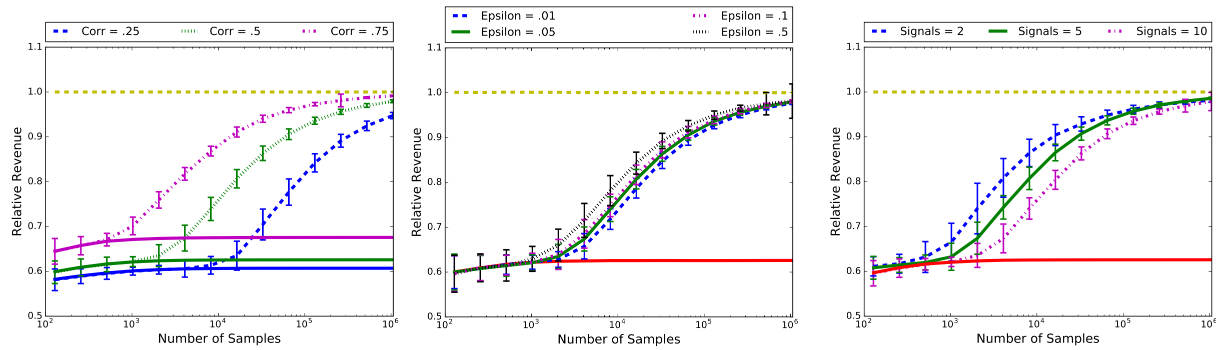
For our experiments, we solve for the optimal ex-post, ϵ-robust, and Bayesian mechanisms given our estimated distribution $\hat{\boldsymbol{\pi}}$ and our ϵ-consistent set. Given that both the optimal ϵ-robust and Bayesian mechanisms can fail to be incentive compatible and/or individually rational due to the difference between the estimated and true distribution, we compute the optimal action for the bidder: either report truthfully, strategically misreport, or do not participate. We then calculate the revenue accordingly.

In Figure 4, we show the performance of the optimal ex-post, robust, and Bayesian mechanisms using our estimated distribution as we increase the number of samples. We report confidence intervals for both the ex-post mechanisms and the robust mechanisms; however for the Bayesian mechanisms, the confidence intervals were off the chart. Figure 4 demonstrates how badly the Bayesian mechanism performs when the distribution is not exactly known. Even after $10,000$ samples from the true distribution, the Bayesian mechanism fails to outperform the ex-post mechanism. By contrast, the optimal ϵ-robust mechanism generates revenue indistinguishable from the ex-post mechanism for low numbers of samples, while significantly outperforming the ex-post mechanism starting at about $10,000$ samples.

In Figures 5a and 5b, we vary correlation and ϵ with increasing numbers of samples. As the bidder type and external signal are more highly correlated, the ϵ-robust mechanism requires fewer samples to perform well, Figure 5a. Also, we see that the ϵ-robust mechanism is not very sensitive

Figure 5 The performance of the robust and ex-post mechanisms using the estimated distribution. All revenue is scaled by the full social surplus, which is denoted as 1. Shown are the 95% confidence intervals for the robust mechanism. For any variable not explicitly shown the following values were used: Correlation = .5, ϵ = .05

(a) Number of Samples versus Correlation. (b) Number of Samples versus ϵ. (c) Number of Samples versus Signal Space.



to the choice of ϵ, Figure 5b, a fact that we attribute to being overly cautious in requiring all elements of the distribution to be in the bounded intervals.

In Figure 5c, we bin some of the external signals together in order to explore the trade-off between estimating a lower dimensional distribution and constructing a mechanism over the full information. Specifically, for the Signals $= 2$ case we put all of $\omega = \{1, ..., 5\}$ into one bin and $\omega = \{6, ..., 10\}$ to a second bin. Note the true signal still has 10 values, we are just binning the observed signal. We find that for a low number of samples, we do much better by binning the external signal, but, while difficult to see on the plot, at higher numbers of samples, it is better to use the full distribution.

Note that we consider the results here to be lower bounds on the performance of optimal ϵ-robust mechanisms. We assume a completely uninformative prior, increasing the required sample size. Further, we have used a naïve distribution estimation procedure, so there is likely significant room to improve upon the estimation.

7. Conclusion

In this work, we have presented a new mechanism design paradigm, ϵ-robust mechanisms, that takes a non-trivial step away from traditional mechanism design paradigms. Specifically, we allow for the traditional constraints of incentive compatibility and individual rationality to be probabilistically violated, and in return, we are able to compute these mechanisms efficiently and guarantee nearly optimal performance when the estimate is sufficient, even when the set of possible distributions is not only infinite but consists of all possible distributions. We also demonstrate that this mechanism design procedure makes efficient use of samples from the underlying distribution. More generally, this class of mechanisms naturally spans the distance between traditional ex-post mechanisms, a setting we can replicate with $\epsilon = 0$, and Bayesian mechanisms, $\epsilon = 1$. Therefore, we effectively parameterize the design of Bayesian mechanisms in uncertain settings.

Since we are looking at the case where full surplus extraction as revenue is possible, it is essential that we can characterize when full surplus extraction as revenue is possible. Therefore, we significantly extend the existing results in the correlated mechanism design literature by providing the first necessary and sufficient conditions for full surplus extraction as revenue. These results suggest that Bayesian incentive compatible mechanisms are likely to lead to significantly higher revenue when the external signal for a bidder is of limited dimension. Our mechanism design procedure incorporates these insights by using robust incentive compatibility as well as robust individual rationality.

We have also presented the extremes of learning mechanisms for settings with correlated bidder distributions. Theorem 5 and Corollary 2 suggest that learning optimal mechanims is doomed in the worst case. Therefore, we designed our class of mechanisms to do as well as traditional ex-post mechanisms in the case where the degree of separation between beliefs is low, since under large uncertainty, the constraint set for the linear program will include nearly all of the constraints for traditional mechanism design.

There is work to be done to analyze the convergence of this ϵ-robust mechanism design procedure to settings where full surplus extraction as revenue is not possible. However, this would require a better understanding of the optimal mechanism under full knowledge of the distribution, which is currently an open question. We hope that our results characterizing necessary and sufficient full surplus extraction conditions may point to a better understanding of the optimal mechanism when our conditions fail. Specifically, it remains an open question as to whether the optimal mechanism is *deterministic* (i.e. the allocation probability is always 0 or 1), and if there exists a way to extend the virtual valuation function approach of Myerson (1981) to a correlated valuation setting that would allow for the simple computation of the allocation function. If these questions are resolved, then it should be possible to better analyze the performance of ϵ-robust mechanisms as the estimate of a set of beliefs that do not permit full surplus extraction as revenue is made more precise.

An area of particular interest for future research is applying these techniques to the problem of budget balanced, socially efficient mechanisms. As the well known Myerson-Satterthwaite impossibility result (Myerson and Satterthwaite 1983) states, it is generally impossible to have strong budget balanced, socially efficient mechanisms. However, in a correlated valuation setting, there is a generic condition that states that strong budget balanced, socially efficient mechanisms are possible (Kosenok and Severinov 2008), but the mechanisms are, much like revenue maximizing mechanisms, highly dependent on the mechanism designer's precise knowledge of the true distribution and a common prior among bidders. However, the results from this work suggest that there are likely to be computationally feasible robust mechanisms that approximately achieve budget balance and social efficiency. This would lead to new areas of incentive compatible distributed systems, such as federated server farms, where a group shares resources in an efficient manner without any money being transferred out of the system. We are currently exploring questions in this direction.

Appendix. Additional Proofs

Additional Proof for Section 3

Proof of Theorem 4. Let the valuations, $v(\theta)$ and the marginal distribution, $\boldsymbol{\pi}_\theta$, be defined as in Example 1. Let $\boldsymbol{\pi}(\omega_L|\theta) = (2^{k(\theta-1)} - 1)/(2^{k(|\Theta|-1)} - 1)$, where $k > 1$. Then, if $G(\boldsymbol{f}) = -2((2^{k(|\Theta|-1)} - 1)\boldsymbol{f}(\omega_L) + 1)^{-1/k}$, $G(\boldsymbol{\pi}(\cdot|\theta)) = -v(\theta)$. Note that $G''(\boldsymbol{f}) > 0$ for $k > 1$, so G is convex.

Since G is convex, the Bayesian IC mechanism extracts full surplus, and full surplus is $|\Theta| + 1$. Also, there does not exist a linear function H such that $H(\boldsymbol{\pi}(\cdot|\theta)) = -v(\theta)$, so the optimal ex-post IC mechanism will not extract full surplus.

Note that for an ex-post IC, ex-interim IR mechanism, given ω, it can be shown by direct application of ex-post IC that for all $\theta \in \Theta$, $p(\theta+1,\omega) \geq p(\theta,\omega)$. Further, for all $\theta \in \Theta, \omega \in \Omega$ such that $p(\theta,\omega) = 1$, $x(\theta,\omega) = x(\omega)$, again by direct application of ex-post IC. Finally, for any optimal ex-post IC mechanism, $p(|\Theta|,\omega) = 1$ for all $\omega \in \Omega$.

Let (p^*, x^*) be an optimal ex-post IC mechanism. Suppose that for $\omega = \omega_H$, there exists a $\theta' \in \Theta$ such that $p^*(\theta',\omega_H) < 1$ and $p^*(\theta'+1,\omega_H) = 1$. Also, by ex-post IC and the assumed optimality, $v(\theta'+1) - x^*(\omega_H) = v(\theta'+1)p^*(\theta',\omega_H) - x^*(\theta',\omega_H)$, because if not $p^*(\theta,\omega_H)$ and $x^*(\theta',\omega_H)$ could be increased, increasing expected revenue and violating the optimality of the mechanism. Therefore, $x^*(\omega_H) = v(\theta'+1)(1 - p^*(\theta',\omega_H)) + x^*(\theta',\omega_H)$ Define a new mechanism (p', x') such that $p'(\theta',\omega_H) = 1$, $x'(\theta',\omega_H) = x^*(\theta',\omega_H) + v(\theta')(1 - p^*(\theta',\omega_H))$, $x'(\omega_H) = x'(\theta',\omega_H)$, and for all other (θ,ω), $p'(\theta,\omega) = p^*(\theta,\omega)$ and $x'(\theta,\omega) = x^*(\theta,\omega)$. Therefore, $x'(\theta',\omega_H) - x^*(\theta,\omega_H) = v(\theta')(1 - p(\theta',\omega_H))$ and $x'(\omega_H) - x^*(\omega_H) = (v(\theta') - v(\theta'+1))(1 - p(\theta',\omega_H))$. Note that (p', x') is ex-post IC and ex-interim IR.

Then, the difference in expected revenue between mechanism (p', x') and (p^*, x^*) is

$$\sum_{\theta,\omega} \boldsymbol{\pi}(\theta,\omega)x'(\theta,\omega) - \sum_{\theta,\omega} \boldsymbol{\pi}(\theta,\omega)x^*(\theta,\omega)$$

$$= \sum_{\theta=\theta'+1}^{|\Theta|} \frac{1}{2^{\theta}} \boldsymbol{\pi}(\omega_H|\theta)(x'(\omega_H) - x^*(\omega_H)) + \frac{1}{2^{\theta'}} \boldsymbol{\pi}(\omega_H|\theta')(x'(\theta',\omega_H) - x^*(\theta',\omega_H))$$
$$= - \sum_{\theta=\theta'+1}^{|\Theta|} \frac{1}{2^{\theta}} \boldsymbol{\pi}(\omega_H|\theta) 2^{\theta'}(1 - p^*(\theta',\omega_H)) + \frac{1}{2^{\theta'}} \boldsymbol{\pi}(\omega_H|\theta') 2^{\theta'}(1 - p^*(\theta',\omega_H)) > 0$$

Therefore, (p^*, x^*) is not optimal, which is a contradiction. This implies that for an optimal ex-post IC mechanism (p^*, x^*) for all $\theta \in \Theta$, $p^*(\theta, \omega_H) = 1$, and by ex-interim IR applied at $\theta = 1$, $x^*(\omega_H) = 2$. Also, since for all $\theta' \in \Theta \backslash \{|\Theta|\}$, $\pi(|\Theta|, \omega_L) v(|\Theta|) = 2 > \sum_{\theta=1}^{|\Theta|-1} \pi(\theta, \omega_L) v(\theta')$, ex-post IR must bind for $\theta = |\Theta|$, which implies $x^*(|\Theta|, \omega_L) = 2^{|\Theta|}$. Further, it is trivial to verify that for all $\theta \in \Theta$, $p^*(\theta, \omega_L) = 1$ and $x^*(\theta, \omega_L) = x^*(|\Theta|, \omega_L)$ is ex-post IC and ex-interim IR. Therefore, the mechanism (q, m) where for all $\theta \in \Theta$ and $\omega \in \Omega$, $p(\theta, \omega) = 1$, $x(\omega_L) = 2^{|\Theta|}$, and $x(\omega_H) = 2$ is an optimal ex-post IC mechanism.

For sufficiently large k, it is easy to verify that $\sum_{\theta \in \Theta \backslash \{|\Theta|\}} (1/2^{\theta}(2^{|\Theta|} \boldsymbol{\pi}(\omega_L|\theta) + 2\boldsymbol{\pi}(\omega_H|\theta))) < 2$. Choose k such that this is true. Then, the expected revenue due to an optimal ex-post IC mechanism is given by:

$$\sum_{\theta,\omega} \boldsymbol{\pi}(\theta,\omega) x(\theta,\omega) = \sum_{\theta \in \Theta \backslash \{|\Theta|\}} \frac{1}{2^{\theta}} \left(2^{|\Theta|} \boldsymbol{\pi}(\omega_L|\theta) + 2\boldsymbol{\pi}(\omega_H|\theta)\right) + (1/2^{|\Theta|-1}) 2^{|\Theta|} < 4.$$

Therefore, the optimal Bayesian IC mechanism has revenue of $|\Theta| + 1$ and the optimal ex-post IC mechanism has revenue less than 4. □

Additional Proofs for Section 5

Proof of Lemma 2. First, note that by Assumption 1, for all $\theta \in \Theta$, there exists $\{\alpha_k\}_{k=1}^{|\Omega|}$ and an affinely independent set of vectors $\{\boldsymbol{\pi}_k(\cdot|\theta_k)\}_{k=1}^{|\Omega|}$, where $\alpha_k \in (0,1)$ and $\boldsymbol{\pi}^*(\cdot|\theta) = \sum_{k=1}^{|\Omega|} \alpha_k \boldsymbol{\pi}_k(\cdot|\theta_k)$. The set of affinely independent points $\{\boldsymbol{\pi}_k(\cdot|\theta_k)\}_k$ define a simplex in $\mathbb{R}^{|\Omega|}$, and the lth face of the simplex, where $l \in \{1, ..., |\Omega|\}$, is the set of points denoted by $\sum_{k \neq l} \alpha'_k \boldsymbol{\pi}_k(\cdot|\theta_k)$ such that $\sum_{k \neq l} \alpha'_k = 1$ and $\alpha'_k \in [0,1]$. The distance from the distribution $\boldsymbol{\pi}^*(\cdot|\theta)$ to any point on the lth face is:

$$\min_{\alpha'_k} ||\boldsymbol{\pi}^*(\cdot|\theta) - \sum_{k \neq l} \alpha'_k \boldsymbol{\pi}_k(\cdot|\theta_k)|| = \min_{\alpha'_k} ||\sum_{k=1}^{|\Omega|} \alpha_k \boldsymbol{\pi}_k(\cdot|\theta_k) - \sum_{k \neq l} \alpha'_k \boldsymbol{\pi}_k(\cdot|\theta_k)||$$
$$= \min_{\alpha'_k} ||\alpha_l \boldsymbol{\pi}_l(\cdot|\theta_l) - \sum_{k \neq l} (\alpha_k - \alpha'_k) \boldsymbol{\pi}_k(\cdot|\theta_k)|| > 0.$$

The last inequality is due to $\alpha_l \neq 0$ (Assumption 1) and affine independence. Let $\epsilon' > 0$ be the minimum such distance for all $\theta \in \Theta$ and all faces of the simplex.

Define $\epsilon_{\min} = \frac{\epsilon'}{2}$. Let $\boldsymbol{\pi}$ be a distribution over Ω where $||\boldsymbol{\pi} - \boldsymbol{\pi}^*(\cdot|\theta)|| < \epsilon_{\min}$, for some $\theta \in \Theta$. Let $\{\boldsymbol{\pi}_k(\cdot|\theta_k)\}_k$ define the simplex that contains $\boldsymbol{\pi}^*(\cdot|\theta)$. Therefore, the distance from $\boldsymbol{\pi}$ to any face of the simplex is at least $\epsilon_{\min}$ by an application of the triangle inequality. Let $\boldsymbol{z} \in \mathbb{R}^{|\Omega|}$ be a unit vector such that $\sum_\omega z(\omega) = 0$. Since a simplex is a closed and bounded set, there exists some face of the simplex which we will denote as the lth face, such that for some $\{\alpha'_k\}_{k\neq j}$ and some $\epsilon \geq \epsilon_{\min} > 0$:

$$\boldsymbol{\pi} - \epsilon\boldsymbol{z} = \sum_{k\neq l} \alpha'_k \boldsymbol{\pi}_k(\cdot|\theta_k).$$

Let $\boldsymbol{\pi}_j(\cdot|\theta_j)$ be a vertex of that face such that

$$\left(\boldsymbol{\pi}_j(\cdot|\theta_j) - \sum_{k\neq l} \alpha'_k \boldsymbol{\pi}_k(\cdot|\theta_k)\right) \bullet \boldsymbol{z} \leq 0.$$

This must exist by virtue of the face being a segment of a hyper-plane. Then:

$$\begin{aligned}
\epsilon_{\min} \leq \epsilon = \epsilon\boldsymbol{z} \bullet \boldsymbol{z} &= \left(\boldsymbol{\pi}(\cdot) - \sum_{k\neq l} \alpha'_k \boldsymbol{\pi}_k(\cdot|\theta_k)\right) \bullet \boldsymbol{z} \\
&= \left(\boldsymbol{\pi}_j(\cdot|\theta_j) - \sum_{k\neq l} \alpha'_k \boldsymbol{\pi}_k(\cdot|\theta_k)\right) \bullet \boldsymbol{z} + (\boldsymbol{\pi} - \boldsymbol{\pi}_j(\cdot|\theta_j)) \bullet \boldsymbol{z} \\
&\leq (\boldsymbol{\pi} - \boldsymbol{\pi}_j(\cdot|\theta_j)) \bullet \boldsymbol{z}
\end{aligned}$$

□

Proof of Lemma 3. Let $\epsilon_{\min} > 0$ be defined as in Lemma 3. By the definition of converging sequences of distributions (Definition 14), there exists a $T \in \mathbb{N}$ such that for all $\theta \in \Theta$ and $\boldsymbol{\pi}_{i^*}(\cdot|\theta) \in \{\boldsymbol{\pi}_i(\cdot|\theta)\}_{i=T}^{\infty}$, $||\boldsymbol{\pi}_{i^*}(\cdot|\theta) - \boldsymbol{\pi}^*(\cdot|\theta)|| \leq \epsilon_{\min}$. Since there are a finite number of distributions such that $i^* < T$, choose $M_{i^*<T} = \max_{i^*<T,\theta,\omega} |x(\theta, \boldsymbol{\pi}_{i^*}, \omega)|$.

Therefore if payments are not bounded, for any $M' > 0$, there must exist some $\boldsymbol{\pi}_{i'} \in \{\boldsymbol{\pi}_i\}_{i=T}^{\infty}, \theta' \in \Theta$, and $\omega' \in \Omega$ such that $x(\theta', \boldsymbol{\pi}_{i'}, \omega') > M'$ or $x(\theta', \boldsymbol{\pi}_{i'}, \omega') < -M'$.

First, we will consider the case where $x(\boldsymbol{\pi}_{i'}, \theta', \omega') < -M'$. Denote the type $\theta \in \Theta$ with the minimum marginal cost by π_{min}, i.e. $\pi_{min} = \min_\theta\{\pi_\theta\}$. Without loss of generality, we can assume that $\pi_{min} > 0$. Note that the expected revenue generated for any type θ must be bounded from below by $-v(|\Theta|)/\pi_{\min}$ if the mechanism guarantees non-zero expected revenue. This is because the maximum amount of expected revenue for any type can be at most $v(|\Theta|)$ or individual rationality will not be satisfied, and if expected revenue for any type is less than $-v(|\Theta|)/\pi_{\min}$, it is not possible to make up the revenue from other types. Further, this implies that in order for the mechanism to generate non-negative revenue, the bidder's expected utility for any type must be less than $v(|\Theta|) + v(|\Theta|)/\pi_{\min}$. Therefore, set

$$M' = \frac{v(|\Theta|)}{\pi_{\min}} + \frac{(1 - \epsilon_{\min})(2v(|\Theta|) + \frac{v(|\Theta|)}{\pi_{\min}} + 1)}{\epsilon_{\min}}$$

Then, the magnitude of the gradient of the hyper-plane defined by the affine combination of $x(\theta', \boldsymbol{\pi}_{i'}, \omega)$ for all $\omega \in \Omega$ must be at least:

$$||\nabla \boldsymbol{x}(\theta', \boldsymbol{\pi}_{i'}, \cdot)|| \geq \frac{(-\frac{v(|\Theta|)}{\pi_{\min}} + M')}{(1-\epsilon_{\min})} = \frac{2v(|\Theta|) + \frac{v(|\Theta|)}{\pi_{\min}} + 1}{\epsilon_{\min}}$$

Let $\boldsymbol{z} \in \mathbb{R}^{|\Omega|}$ with $\sum_\omega z(\omega) = 0$ be the direction of the gradient of the hyper-plane defined by the lottery in the plane of the $|\Omega|$-simplex. Then by Lemma 2, there exists a $\boldsymbol{\pi}_j(\cdot|\theta_j)$ such that $(\boldsymbol{\pi}_{i'}(\cdot|\theta') - \boldsymbol{\pi}_j(\cdot|\theta_j)) \bullet \boldsymbol{z} > \epsilon_{\min}$. Then:

$$\begin{aligned}
\sum_\omega \pi_j(\omega|\theta_j) U(\theta_j, \boldsymbol{\pi}_j, \theta_j, \boldsymbol{\pi}_j, \omega) \geq \sum_\omega \pi_j(\omega|\theta_j) U(\theta_j, \boldsymbol{\pi}_j, \theta', \boldsymbol{\pi}_{i'}, \omega) \qquad \text{(by IC)} \\
&\geq \sum_\omega \pi_j(\omega|\theta_j) U(\theta_j, \boldsymbol{\pi}_j, \theta', \boldsymbol{\pi}_{i'}, \omega) - \sum_\omega \pi_{i'}(\omega|\theta') U(\theta', \boldsymbol{\pi}_{i'}, \theta', \boldsymbol{\pi}_{i'}, \omega) \quad \text{(by IR)} \\
&\geq \sum_\omega (\pi_{i'}(\omega|\theta') - \pi_j(\omega|\theta_j)) x(\theta', \boldsymbol{\pi}_{i'}, \omega) - v(|\Theta|) \\
&= (\boldsymbol{\pi}_{i'}(\cdot|\theta') - \boldsymbol{\pi}_j(\cdot|\theta_j)) \cdot \boldsymbol{z} ||\nabla \boldsymbol{x}(\theta', \boldsymbol{\pi}_{i'}, \cdot)|| - v(|\Theta|) \\
&\geq \epsilon_{\min} ||\nabla \boldsymbol{x}(\theta', \boldsymbol{\pi}_{i'}, \cdot)|| - v(|\Theta|) \\
&\geq v(|\Theta|) + v(|\Theta|)/\pi_{\min} + \epsilon_{\min}
\end{aligned}$$

Therefore, the seller cannot earn non-negative expected revenue for type $(\boldsymbol{\pi}_j, \theta_j)$, a contradiction.

It is straightforward to show that the combination of individual rationality and payments being bounded from below by $-\max\{M_{i^*<T}, M'\}$ implies that all payments must be bounded from above. We omit the details. Denote this upper bound by M''.

Therefore, let $M = \max\{M_{i^*<T}, M', M''\}$, and all payments are bounded by M. □

Additional Proofs for Section 6

Proof of Lemma 6. Note that the payments varying by $\omega \in \Omega$ only affects the robust IC constraint (equation (7)). This is because the objective is only affected by the expected payment given $\theta \in \Theta$ and the marginal probability, $\hat{\pi}(\theta)$. Moreover, the robust IR constraint (equation (6)), similarly, only depends on the expected payment for any given conditional distribution.

Therefore, the only role of conditioning payments on ω is to separate types to ensure that the bidder reports his type truthfully. Note that for any subset of types $\Theta' \subset \Theta$ such that $|\Theta'| = \max\{|\Theta|, |\Omega|\}$ and for any $\boldsymbol{\pi}' \in \mathcal{P}(\hat{\boldsymbol{\pi}})$, there exists a hyperplane through the points $\{(\boldsymbol{\pi}'(\cdot|\theta), v(\theta))\}_{\theta \in \Theta'}$, by the assumption of γ-separation. Moreover, there exists a hyperplane such that the gradient is bounded by $\frac{|v(|\Theta|) - v(1)|}{\gamma} < \frac{v(|\Theta|)}{\gamma}$, again by the definition of γ-separation. Therefore, a lottery defined by a hyperplane with a gradient of $\frac{v(|\Theta|)}{\gamma}$ is sufficient to separate any subset of types.

Moreover, if for some $\theta \in \Theta$ the expected payment for a lottery $\boldsymbol{x}(\theta, \cdot)$ is negative for every possible distribution $\boldsymbol{\pi} \in P(\Omega)$, then the expected revenue for the mechanism can be increased by raising the payments for type θ until there is some distribution over Ω such that the expected payment is nonnegative. Therefore, there exists an optimal mechanism, $(\boldsymbol{p}, \boldsymbol{x})$, such that for every lottery defining hyperplane, the hyperplane's gradient is bounded by $v(|\Theta|)/\gamma$, and the value of the hyperplane is at least 0 somewhere in the set of distributions over Ω. Similarly, the value of the hyperplane must be less than or equal to $v(|\Theta|)$ for some distribution over Ω or the lottery is not individually rational for any bidder type. Therefore for all $\theta \in \Theta$ and $\omega \in \Omega$, the maximum payment is $-\frac{v(|\Theta|)}{\gamma} \leq x(\theta, \omega) \leq \frac{v(|\Theta|)}{\gamma} + v(\Theta) \leq \frac{2v(|\Theta|)}{\gamma}$, since $\gamma \leq 1$. $\square$

Proof of Lemma 7. Construct a mechanism, $(\boldsymbol{p}', \boldsymbol{x}')$, as in the proof of Theorem 6, for the distribution $\boldsymbol{\pi}^*$ and δ. By Lemma 6 and Theorem 6, this mechanism guarantees that the revenue for any distribution $\boldsymbol{\pi} \in \mathcal{P}(\hat{\boldsymbol{\pi}})$ is within $\frac{k}{3|\Omega|}$ of the optimal revenue, and the mechanism is robust IR. However, the mechanism does not necessarily satisfy robust incentive compatibility.

Choose type $\theta \in \Theta$, let $\Theta'(\theta) \subset \Theta$ be the set of types that maximizes utility for some $\boldsymbol{\pi}(\cdot|\theta) \in \mathcal{P}(\hat{\boldsymbol{\pi}}(\cdot|\theta))$, i.e.

$$\Theta'(\theta) = \left\{ \theta' \in \Theta \,\middle|\, \exists \boldsymbol{\pi}(\cdot|\theta) \in \mathcal{P}(\hat{\boldsymbol{\pi}}(\cdot|\theta)) \, s.t. \, \forall \hat{\theta} \in \Theta, \sum_{\omega} \pi(\omega|\theta) U(\theta, \theta', \omega) \geq \sum_{\omega} \pi(\omega|\theta) U(\theta, \hat{\theta}, \omega) \right\}.$$

Then, construct a new set of payments $\boldsymbol{x}^*$ such that 1) for all $\boldsymbol{\pi}(\cdot|\theta) \in \mathcal{P}(\cdot|\theta)$ and $\theta' \in \Theta'(\theta)$, $\sum_{\omega} \pi(\omega|\theta) x^*(\theta, \omega) \leq \sum_{\omega} \pi(\omega|\theta) x'(\theta', \omega)$, and 2) minimizes

$$\beta = \max_{\{\pi_\omega \mid \sum_\omega \pi_\omega = 1\}} \min_{\theta' \in \Theta'} \sum_i \pi_\omega (x'(\theta', \omega) - x^*(\theta, \omega))$$

i.e., β is the maximum distance from the minimum expected payment for any lottery and any distribution. β must be positive by the first condition. The first condition also ensures that it is indeed incentive compatible for the bidder to report θ, and the second condition ensures that it is the largest set of payments such that this is true. Moreover, the value of β is at most $\frac{k}{3|\Omega|}$, and the utility for $\boldsymbol{\pi}(\cdot|\theta) \in \mathcal{P}(\hat{\boldsymbol{\pi}}(\cdot|\theta))$ is at most $\frac{k}{3|\Omega|}$. Note that for each $\theta' \in \Theta'$ there is a hyperplane that is defined by $x(\theta', \cdot)$, and this set of hyperplanes intersects inside the set $\mathcal{P}(\hat{\boldsymbol{\pi}}(\cdot|\theta))$. Given than the minimum of an intersecting set of hyperplanes defines a concave function, one can always construct a hyperplane that is less than or equal to the concave function for a given set, $\mathcal{P}(\hat{\boldsymbol{\pi}}(\cdot|\theta))$, and the difference between the concave function and the hyperplane is non-increasing outside of the set. Therefore, the maximum occurs within the convex set, and the value of the hyperplane at the boundaries of the convex set are greater than or equal to $v(\theta) - \frac{k}{3|\Omega|}$, by the construction of $\boldsymbol{x}'$, and the fact that $\boldsymbol{x}^*$ minimizes β.

However, there may be another $\theta^* \in \Theta$ that now chooses to report θ in order to have payments $x^*(\theta, \omega)$. Construct new payments $x_2^*(\theta^*, \omega) = x_2^*(\theta, \omega) = x^*(\theta, \omega) - \frac{k}{3|\Omega|}$ for $\omega \in \Omega$ and $x_2^* = x^*(\theta', \omega)$ for $\theta' \neq \theta$. Then, by an identical argument to Theorem 6, θ^* will optimally report θ^*. Now, check to see if there exists another type θ^{**} that now prefers to misreport type θ. If so, repeat the above procedure. Note that this can happen for at most $|\Omega|-1$ types, by the definition of γ-separated and the necessity for all types must be near the hyperplane defined by $\boldsymbol{x}^*(\theta,\cdot)$. Every type that faces this set of payments, has a utility less than $k\left(\frac{|\Omega|-1}{3|\Omega|}\right)$.

Now choose another $\theta'' \in \Theta$ and repeat. However, all types that would have optimally misreported as θ in the previous round, will never choose to misreport as θ'', since they must lie in a different hyperplane then the payments defined for θ''. Denote the final set of payments as $\boldsymbol{x}$.

Therefore, there exists an allocation $\boldsymbol{p}'$ and set of payments $\boldsymbol{x}$ such that the utility for all types is less than $k\left(\frac{|\Omega|-1}{3|\Omega|}\right)$ and the item is always allocated. This implies the revenue generated by the mechanism at $\boldsymbol{\pi}^*$ is at within $k\left(\frac{|\Omega|-1}{3|\Omega|}\right)$ of the optimal revenue, R. Moreover, using this mechanism, the revenue achievable for any $\boldsymbol{\pi} \in \mathcal{P}(\hat{\boldsymbol{\pi}})$, including $\hat{\boldsymbol{\pi}}$, is bounded by shifting all probability mass from the largest payment to the smallest payment, a loss of revenue of less than $\delta\left(\frac{3v(|\Theta|)}{\gamma}\right) \leq \frac{k}{2|\Omega|}$ by Lemma 6. Thus, the optimal robust mechanism at $\mathcal{P}(\hat{\boldsymbol{\pi}})$ has an objective value of at least $R - k\left(\frac{|\Omega|-1}{3|\Omega|}\right) - \frac{k}{2|\Omega|}$.

The last piece of the proof is to show that for any mechanism that does nearly optimally for $\hat{\boldsymbol{\pi}}$ will also do nearly optimally for all $\boldsymbol{\pi} \in \mathcal{P}(\hat{\boldsymbol{\pi}})$. However, the worst that could happen is, again, all probability mass could shift to the worst possible payment from the best possible payment, which the difference between the two is bounded by $\frac{3v(|\Theta|)}{\gamma}$ by Lemma 6. Therefore, the maximum loss due to mis-specifying the objective is $\delta\left(\frac{3v(|\Theta|)}{\gamma}\right) \leq \frac{k}{2|\Omega|}$. Therefore, for all $\boldsymbol{\pi} \in \mathcal{P}(\hat{\boldsymbol{\pi}})$, the optimal robust mechanism for $\mathcal{P}(\hat{\boldsymbol{\pi}})$ achieves a revenue of $R - k\left(\frac{|\Omega|-1}{3|\Omega|}\right) - \frac{k}{2|\Omega|} - \frac{k}{2|\Omega|} \geq R - k$ on $\boldsymbol{\pi}$. $\square$

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